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THE TRIPLE SYSTEM OF 13 CETI

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ABSTRACT

The system of 13 Ceti, in which three components are now known with certainty, has always presented unusual difficulties in observation and interpretation. Several orbits, visual as well as spectroscopic, have been derived—none as yet entirely satisfactory—and many suggestions have been made concerning various possible sources of perturbation.

It is shown in the present paper that the orbit derived by Maggini from a large number of interferometer measures does not obey the law of areas, and from a rediscussion of the entire material a new visual orbit is derived which accords closely with Aitken's original one. The various suggestions and explanations offered for the observed perturbations are discussed, and it is shown that most of these are quantitatively unreasonable, while furthermore it appears that in most cases the perturbations were a direct result of the use of too many uncertain decimals.

The spectroscopic orbit is derived anew from all the available material and it is found to possess a definite, though small, eccentricity, while furthermore a large advance of periastron appears to be indicated. This is tentatively explained as due to the precession operating in the spectroscopic system owing to the large tidal deformation of the two components.

Finally, by making plausible assumptions concerning the orbital planes, values are derived for the absolute dimensions of the spectroscopic system and for the individual masses of the three components now known.

1. The star 13 Ceti (α 0^h30^m,$\delta - 4°09'$,1900.0,5.6,6.4,Go) was discovered to be a close visual double by Hough in 1887. The brighter component was found to be a spectroscopic binary by Frost[1] in 1907, while orbits were first calculated by Aitken[2] (visual) in 1907, and by Fox (spectroscopic)[3] in 1908. Paraskevopoulos[4] and

[1] *Astrophysical Journal*, **25**, 60, 1907.

[2] *Publications of the Lick Observatory*, **12**, 5–6, 1912.

[3] *Astrophysical Journal*, **27**, 375, 1908. [4] *Ibid.*, **52**, 110, 1920.

Pogo[5] further investigated the spectroscopic system, and Maggini[6] derived a new orbit for the visual pair from a long series of interferometer measures. The system is one of great complexity and presents unusual difficulties in the observations, especially in the spectroscopic observations where only one spectrum, that of the brighter component of the spectroscopic pair, appears with certainty, though it may be disturbed by being blended with that of the fainter visual component. In an effort to explain the many discrepancies present in the observations, numerous suggestions have been made. In discussing the orbit of the spectroscopic pair, Paraskevopoulos claimed to have found clear evidence of perturbations in it by the fainter, more massive, visual component during periastron passage. The material used consisted of eight spectrograms taken during the opposition of 1906, ten taken in 1907, eleven in 1908, and ten in 1912. The first two sets were combined into one group, and for each of the three groups a separate set of elements was derived. The ten observations in the last set were subjected to the method of least squares and "definitive" elements were calculated.

Paraskevopoulos then claimed that a comparison of these three sets of elements showed clearly that the spectroscopic period is longer during visual periastron than during apastron, that the value of $a \sin i$ varies, owing to changes in either a or i or both, and that the motion of the line of apsides is well established. The three systemic velocities which hold for different positions in the visual orbit were then used to calculate the true systemic velocity $\gamma_0 = 7.134$ km/sec., and a constant, which involves the parallax and the visual mass ratio. Finally, four more observations made in 1919 were adduced and compared with the velocities calculated from the 1912 elements; the resulting differences O−C, which run as high as 30 km/sec., were taken to show "very clearly the variation of the elements of the orbit."

Pogo, from much more extensive spectroscopic material, claims that the spectroscopic orbit is circular, and attributes such irregularities as are found to exist in the observations to changes in the atmospheres of the stars or to disturbing effects arising from the spectroscopic duplicity of the fainter visual component. Maggini sug-

[5] *Ibid.*, **68**, 116, 1928. [6] *Astronomische Nachrichten*, **233**, 97, 1928.

gests that the fainter visual component B has an invisible companion and moves in an orbit with a radius of about 0".03–0".04. Finally, Slavenas,[7] in the course of a photometric investigation, used 13 Ceti as a comparison star and suspects it to be variable by $0^m.2$ with a period of $0^d.4692$.

2. In making these various suggestions and explanations the fundamental point whether they are quantitatively possible or even reasonable seems to have been continuously overlooked.

In the discussion of Paraskevopoulos, thirty-nine observations in all were used to calculate no less than eighteen different unknowns; furthermore, the values O–C tabulated for each observation indicate a mean error in each velocity of about 2.6 km/sec., hence it is evident that not more than three, and certainly not seven places are warranted in $a \sin i$, that the differences in K, $a \sin i$, and e need not be real, and that ω must be exceedingly uncertain. Where the intervals covered by the 1908 and 1912 observations measure only some sixty and thirty-six periods, respectively, the uncertainty in the period may well be as much as $0^d.01$ and hence even the differences in period need not be real. With as small an eccentricity as the present, it is certainly not warranted to give T to $0^d.0001$, especially since no mention has been made as to whether or not the observations have been reduced to the sun. Of the eleven observations in 1908 no less than nine deviate systematically from the curve drawn, while the large values for O–C for the four isolated observations of 1919, instead of "showing clearly" that perturbations exist, would, in the writer's opinion, furnish additional evidence that the calculated elements are uncertain. The least-squares solution which gave $\gamma_0 + 7.134$ would, if its precepts had been followed, yield for the mean error of γ, ± 2.3, and for C, 11.372 ± 4.3.

The evidence presented by Paraskevopoulos does not, therefore, prove that perturbations exist, but indicates rather clearly that the material is insufficient to draw definite conclusions.

It is true that, qualitatively at least, the perturbation of the period is in the right direction. Quantitatively we may, perhaps, make a very rough estimate by borrowing some results from the lunar theory. As a first approximation we may neglect the eccentricity of the

[7] *Ibid.*, p. 125.

spectroscopic orbit, and assume it and the visual orbit to be coplanar. There then is no orthogonal perturbation, and the chief perturbative force active in lengthening the period is the normal component whose average value is

$$-\tfrac{1}{2}k^2S\cdot\frac{r}{R^3},^{8}$$

where, in the lunar theory k is the Gauss constant, S the mass of the sun, r the distance earth–moon, and R the distance earth–sun. The ratio ϵ of this to the attractive force of the earth on the moon is

$$\frac{S}{2(E+M)}\quad\frac{r^3}{R^3};$$

the consequent lengthening of the period, since $1/P\propto\sqrt{\text{mass}}$, would be roughly given by

$$\frac{\Delta P}{P}=\tfrac{1}{2}\epsilon=\frac{S}{4(E+M)}\cdot\frac{r^3}{R^3}.$$

In the case of the sun–moon, $S/(E+M)$ is 3.2×10^5 and $R/r=400$, hence $\Delta P/P=1/800$, which is at least of the right order of magnitude, as the actual lengthening is about one hour, or $1/650$.[9]

As we shall see later, we may reasonably assume in the present case $m_B=1.1$, $m_{A_1+A_2}=1.3$, $r=0.035$ astronomical units, $R=1.29$ A.U. at visual periastron, and 8.5 A.U. at visual apastron. We then find $\Delta P=0.8\times10^{-5}$ days at periastron, and, of course, a negligible quantity at apastron. Paraskevopoulos finds $\Delta P=0.4\times10^{-2}$, or five hundred times as much, and while undoubtedly the simple lunar theory is not applicable in this instance, where the mass ratio A_1/A_2 is about 3:1 instead of 81:1, it is hardly likely that this would increase the effect five hundred fold. A further, more serious objection is that Paraskevopoulos has overlooked the fact that his longest spectroscopic period corresponds to the mean epoch 1912.9, or $0.1P$, after periastron passage. In an orbit as highly eccentric as the present visual one, this gives $R=4.1$ A.U., or nearly half the apastron distance, and hence ΔP only 0.3×10^{-6} days, or $1/15{,}000$ of that

[8] Moulton, *Celestial Mechanics*, p. 349, 1928.

[9] *Ibid.*, p. 348.

claimed by Paraskevopoulos. We are thus forced to the conclusion that the actual perturbations are much too small to be observable.

Pogo's claim that the spectroscopic orbit is circular will be taken up farther on, and shown to be erroneous.

Following up Maggini's suggestion, we find that the semiaxis major of the relative orbit of B_1 and B_2 is at least $0''.07$, or equal to the periastron distance of A and B, hence one would expect severe perturbations to result.

Furthermore, it is seen that although Maggini's ninty-four observations are extremely concordant, and define the apparent ellipse

TABLE I

MAGGINI'S INTERFEROMETER MEASURES

Date	θ	r	Area Per Cent of Ellipse	Δt Years	Areal Constant
1922.95.....	283°.5	0".260	0.157	1.13	0.139
1924.08.....	306.0	.208	.118	1.11	.106
1925.19.....	339.8	.126			
1925.62.....	20.	.060	.076	0.91	.083
1926.00.....	171.	.070			
1926.10.....	190.	.105			
1926.20.....	198.6	.130	.166	0.82	.202
1926.92.....	228.7	.230	.165	0.93	.178
1927.85.....	249.4	.278	0.318	1.99	0.161
1929.84.....	283.5	0.260			

with great precision, while also this apparent ellipse fits the visual observations tolerably well, the law of areas is not obeyed in this ellipse by the interferometer measures themselves. Taking several normal places whose positions appear to be indicated with almost absolute precision from the observations, we find the values shown in Table I for the areas described (expressed as a fraction of the whole ellipse) and for the times consumed.

It is evident that the apparent periastron distance is too small, hence the eccentricity is too large—and this is the chief discrepancy between Maggini's orbit and Aitken's visual one. Also, it must be regrettably admitted that the interferometer measures are apparently subject to large systematic errors. (If Maggini's orbit is abandoned, his suggestion concerning the duplicity of B automatically follows.)

Slavenas' suggestion of a variation in total light of $0^m.2$ with a period of $0^d.4692$ is very hard to fit in with any data at the present time. The length of period suggests either a cluster variable or an eclipsing binary. Since the brightest star of the system is a G dwarf, the first suggestion is ruled out. The spectroscopic period of A is 2.08, hence it would seem that only B could be the eclipsing binary, but if the total light of AB changes by 0.2, then the fluctuation of B alone should surpass $0^m.5$. It should be possible to verify this by visual estimates, but until it is done the eclipsing binary character of B cannot be considered as established.

TABLE II

ELEMENTS OF THE VISUAL ORBIT OF 13 CETI

	Visual		Interferometer	
	Aitken	Present Paper	Maggini	Maggini Corrected
Ω	38°.7	28°.7	2°.5	23°.0
i*	+53.4	+53.0	+52.6	+50.0
ω	66.8	75.0	85.0	65.0
a	0.242	0.249	0.247	0.24
e	0.725	0.736	0.816	0.71
P	6.88	6.93	6.895	6.92
T	1905.27	1919.20	1925.92	1925.94

* The positive sign for i was assumed to be known from spectroscopic observations.

FINAL ELEMENTS

$\Omega = 24°$	$e = 0.73$
$i = +52$	$P = 6.91$ years
$\omega = 73$	$T = 1925.94$
$a = 0''.244$	Angles increasing

Finally, the visual orbit derived by Aitken shows residuals O − C with the observations of as much as 40° in angle.

In view of the generally unsatisfactory state of the spectroscopic as well as the visual system, it has been thought worth while to discuss all available observations anew. In what follows the visual components will be referred to as A and B, the spectroscopic components of A as A_1 and A_2.

3. *Visual orbit.*—Using all available visual observations given in Burnham's and Aitken's catalogues, also a recent observation by Van den Bos, new elements were calculated. In this connection I am much indebted to Mr. Arnold Rosenstein, a graduate student in our department, who performed all the actual calculations. It should be emphasized that his calculations were made entirely without knowledge of Aitken's or Maggini's orbit, and hence the results should be wholly independent. In Table II are given, in order, Aitken's elements, those derived here, Maggini's interferometer orbit, and a set of elements derived from Maggini's observations by roughly reshaping the ellipse to make it fit the law of areas. The values finally adopted for the elements are given below the table; the graphical representation of the observations is shown in Figure 1.

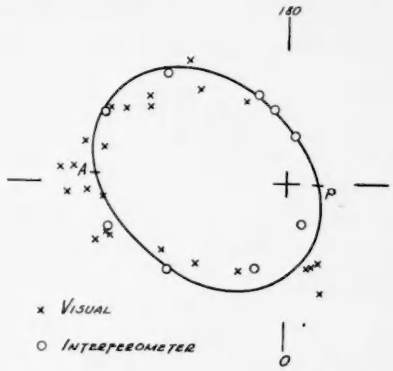

FIG. 1.—The visual orbit of 13 Ceti. Crosses represent visual measures; open circles, normal places derived from Maggini's interferometer measures (cf. Table I).

Two trigonometric values for the parallax are available; viz., 0".048 (S_{51} rel.) and 0".051 (Y_{115} rel). Using the systematic corrections derived by the writer[10] and taking into account the spectroscopic parallax, we would obtain 0".050 for the absolute value of p. Hence the absolute dimensions of the system are

$$a = 4.9 \text{ astronomical units}, \quad q = 1.3 \text{ astronomical units},$$
$$\text{total mass} = 2.4 \odot .$$

4. *Spectroscopic orbit.*—Using the new visual elements, we may now calculate the relative radial velocity of the two components, the systemic velocity of A and the corrections to be applied for light-time at various epochs due to the orbital motion of A around the center of gravity of AB. Actually observed systemic velocities of A are available at many different dates, and may be taken directly

[10] *Proceedings of the National Academy of Sciences*, **16**, 472, 1930.

from Pogo's paper. At this point it is a pleasure to acknowledge the writer's obligation to Dr. Otto Struve for calling attention to this paper, of which the writer had been totally unaware. The much larger spectroscopic material that has become available through Pogo's paper has made it possible to verify the conclusions derived to a much greater extent than before.

If Γ be the true systemic velocity of AB, V the orbital velocity of B with respect to A, γ the observed systemic velocity of A, and β the ratio: mass of B/mass of total system, we have

$$\gamma=\Gamma-\beta \cdot V .$$

Performing the calculations for those epochs for which Pogo has given the observed systemic velocities of A, and which rest on five or more observations,[11] we obtain the data collected in Table III; these are plotted in Figure 2.

TABLE III

t	V	γ	t	V	γ
1906.90	−11	+10.0	1917.06	+ 6	+17.0
07.87	− 6	+11.0	23.72	+ 4	+18.4
08.89	− 1	+ 8.0	25.75	+29	+ 7.0
11.72	−25	− 3.0	26.87	−16	+13.0
12.97	−16	+17.0	27.95	−10	+13.5

The peculiar differences, large in amount and systematic in character, already mentioned by Pogo, are clearly seen. The slope of the points, which defines β, however, is unmistakable, and we may perhaps adopt for the present the graphically derived values

$$\Gamma=+11.0 ; \qquad \beta=0.45$$

and hence the following values for the masses A: 1.27, B: 1.05, the second decimal being added merely for purposes of calculation.

Using β, it is further possible to calculate the corrections for light-time due to the orbital motion of A; making also the usual heliocentric light-time correction, we may now reduce all spectroscopic observations to a uniform system.

[11] *Op. cit.*, Table XI.

All observations given by Paraskevopoulos were first dealt with; the four sets made during the oppositions of 1906, 1907, 1908, and 1912 were reduced separately with a preliminary value of $2^d.082$ for the period, the times of maximum and minimum velocity read off, and these used to calculate a second approximation for the period, viz., $2^d.08186$, with a mean error of about three in the last place. With this all observations were reduced to the mean epoch of all, around J.D. 2418402; furthermore, the velocities were corrected for the calculated systemic velocity of A—that due to its orbital motion, as well as the "true" systemic velocity of AB.

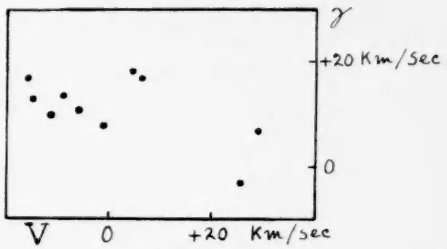

FIG. 2.—Abscissae: relative velocity of B referred to A, as calculated from the visual orbit; ordinates: observed systemic velocities of A, taken from Table III.

The plot of all thirty-nine observations is given in Figure 3. Considering the difficulties and uncertainties inherent in the observations, the velocity-curve is well defined; the four isolated observations made in 1919 have been added, but have not otherwise been used. Planimeter readings give a residual systemic velocity of -2 km/sec., which was to be expected considering the distribution of points in Figure 2.

The outstanding feature of the curve is the great sharpness of minimum as contrasted against the relative flatness at maximum. Coupled with the fact that the curve is practically symmetrical around the ordinates of maximum and minimum velocity, this shows clearly that the orbit is eccentric, and that the longitude of periastron is roughly 180°; T is therefore immediately given as the instant of minimum velocity.

The same procedure cannot be immediately applied to Pogo's forty-seven observations made during 1923–1928 since it is evident here that the systematic correction to systemic velocity is not the same in each year. Since for the present we are concerned only with the shape of the velocity-curve, it is perhaps allowable to determine these corrections empirically. A reasonably good agreement may

be obtained by applying the following corrections to the velocities.

1923–1925	−9 km/sec.
1926	+6
1927–1928	+1

The resultant velocities were then reduced to the mean epoch of all, around J.D. 2424548. Again it is seen that they conform tolerably well to a smooth curve, which is plotted in Figure 4.

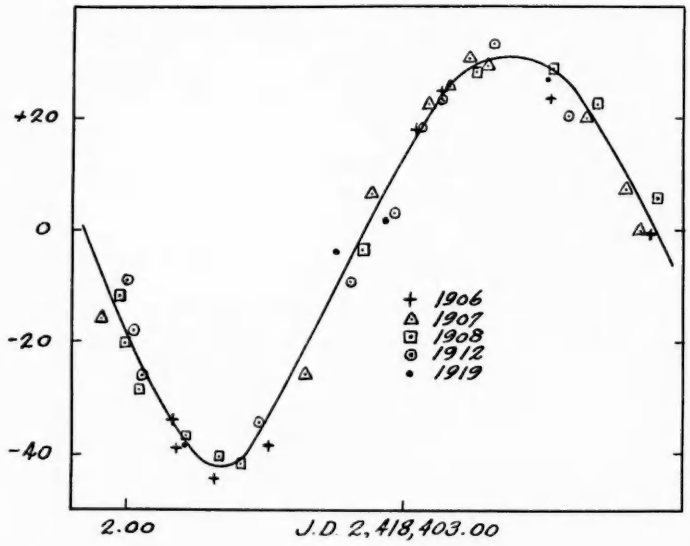

FIG. 3.—The velocity-curve of 13 Ceti A during the years 1906–1912. Abscissae indicate the phase, reduced to the common epoch of J.D. 2418402, after correction for heliocentric and astrocentric light-time had been made. Ordinates indicate the observed velocities, corrected for true systemic velocity of AB and orbital motion of A.

Calculating the elements of the orbits corresponding to Figures 3 and 4 in the usual way, we obtain the data given in Table IV; the system velocities have, of course, been omitted since these now measure only the amount of systematic error in the observations.

The close agreement in the value of e is doubtless accidental, but in any case it is clear that the small values of 0.015 ± 0.018 and 0.022 ± 0.018 obtained by Pogo for the 1924 and 1928 curves are entirely

incompatible with the marked eccentricity of the 1906–1912 curve. Pogo further remarks on the change in ω found by Paraskevopoulos that not only do we "have the case of a circular orbit, not the case of

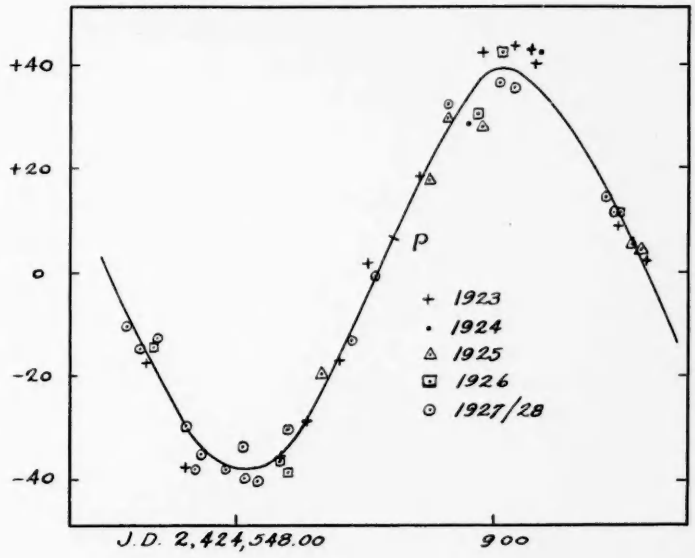

FIG. 4.—The velocity-curve of 13 Ceti A during the years 1923–1928. Abscissae indicate the phase, reduced to the common epoch J.D. 2424548, after correction for heliocentric and astrocentric light-time had been made. Ordinates indicate the observed velocities corrected for orbital motion of A, systemic velocity of AB, and such empirical corrections as appeared necessary to reduce them to the same level.

the motion of the line of apsides" but also that "there is another reason why it would be futile to interpret these discordant results as

TABLE IV

	1906–1912	1923–1928
K	36.8	38.4
e	0.10	0.11
ω	180.	280.
T	J.D. 2418402.33	J.D. 2424548.60
$a \sin i$	1.04×10^6	1.08×10^6
	$P = 2^d.08186$	

a motion of the line of apsides." The reason advanced is that the perturbation produced by the fainter visual companion would

change the direction of the line of apsides by something like 1° in several hundred years, according to a formula derived by Slavenas.[12]

While the present writer agrees with this statement in so far as it applies to the orbits derived by Paraskevopoulos and the perturbation caused by B, the orbit can most certainly not be dismissed as circular at the present time. Furthermore, in the present writer's opinion the systematic uncertainties in the systemic velocities make it futile to apply a least-squares solution in order to calculate the small but meaningless probable errors in e. Finally, blaming any possible motion of the line of apsides on the companion B is looking in the wrong place for there is another far more likely cause for such motion.

Using the masses for A and B determined above, viz., 1.27 and 1.05, respectively, we find, with $P = 2^{d}.08 = 0^{y}.0057$, a (spectr.) = 0.035 astronomical units. Since $a \sin i$ for the absolute orbit of A_1 is 1.06×10^6 km, or 0.0071 A.U., the mass ratio A_1/A_2 cannot be more than 4:1. Making the reasonable assumption that the visual and spectroscopic systems are approximately coplanar, we have sin $i = 0.8$ and a (A_1) = 0.009 A.U., giving a mass ratio of 3:1 in the spectroscopic system and individual masses of 0.95 and 0.32, respectively. Since the spectral class of A_1 is G and its absolute magnitude 4.1, it would thus seem that this brighter component is very similar to the sun, while A_2 might well resemble an M dwarf; hence their diameters should be of the order of 1.5×10^6 and 0.7×10^6 km, respectively, i.e., the separation of their surfaces is only about three to four times the sum of their radii. Presumably the axial rotations of both components are synchronized with their revolution; this would make the ratio φ of the centrifugal force at the equator of the larger star to the attractive force about the same as for the earth. But where the two stars are so close their tidal deformations may well be large, and hence we should theoretically expect a rapid advance of the line of apsides.

Even though all numerical quantities involved are extremely uncertain, it might be of value to see whether the observed motion of the line of apsides appears reasonable. Where the fainter component

[12] *Ibid.*, p. 133.

of the spectroscopic system is presumably of less mass, of much smaller size, and of considerably higher density than the primary, we may perhaps, as a first approximation, treat it as a satellite revolving around an oblate planet. In that case the ratio K of the period of rotation of the line of apsides to the period of orbital revolution is given by

$$K=\frac{a^2}{r^2}\,\frac{1}{\epsilon-\frac{1}{2}\varphi},$$

where a is the semiaxis major of the orbit, r the radius of the larger stars, ϵ its oblateness, and φ the ratio mentioned above. In the present case we have $K=10^4$, $a=5.2\times10^6$ km, $r=7\times10^5$ km, $\varphi=1/300$; hence $\epsilon=1/150$, which appears not unreasonable.

A glance at the data of Table IV shows that periastron has advanced by about 100° during the 2952 periods which have elapsed between the two sets of elements, but this number of 100 may well be uncertain by some 20°–30°. Hence we find for the rate of advance of periastron $0.03°/P$. Since a true anomaly of 100° corresponds, for $e=0.10$ and $P=2^d.08$, to a time interval of $0^d.51$, we finally obtain

Anomalistic period	$2^d.08207$
Sidereal period	$2^d.08189$

With the adopted velocity of +11 km/sec. for the center of mass of the entire system, this gives for the "true" sidereal period, freed from Doppler effect, $P=2^d.08182$.

In order to settle the matter conclusively, it would probably be necessary to obtain a large number of observations during at least two more epochs separated by some ten years. At the next visual apastron passage it might perhaps be possible, by setting the slit of the spectroscope at right angles to the visual position angle, and by using only nights of excellent definition, to obtain some impression of the spectrum of the fainter visual component, and to decide whether this component, too, is perhaps not a spectroscopic binary, which seems not improbable on account of its mass.

5. *Summary.*—Tentatively combining the two spectroscopic orbits into one, we may give the following summing-up of our present knowledge concerning the system 13 Ceti:

R.A. 0:30 Dec. −4:09 5.6 Go, 6.4 $-p=0''.050$
$\mu=0''.41$ in 93°, or $T=39$ km/sec., $V=+11$ km/sec.

Visual Orbit	Spectroscopic Orbit
$P=6.91$ years	$P=2.08189$ days
$a=0''.244=4.9$ astr. units	$a \sin i=1.06\times10^6$ km
$e=0.73$	$e=0.1$
$\Omega=24°$	$K=37$ km/sec.
$i=+52°$	
$\omega=73°$	$\omega=180°.0°.016(t-T)$
$T=1925.94$	$T=$J.D. 2418402.33
Angles increasing	

MASSES: $A_1=0.95$, $A_2=0.32$, $B=1.05$

MINNEAPOLIS, MINN.
June 14, 1933

OPACITY FORMULAE AND STELLAR LINE INTENSITIES[1]

By HENRY NORRIS RUSSELL[2]

ABSTRACT

Theory.—Formulae for the general opacity of a stellar atmosphere are reviewed, especially Pannekoek's approximations. General equations are derived (§ 5) for the numbers of neutral and ionized atoms and the optical depth in an atmosphere containing *two partially ionized elements* and *others not ionized*, and simplified by the introduction of functions *f*, which are tabulated (§ 8). Solutions in which only *one component is supposed to be partially ionized*—the rest being *completely ionized* or *completely neutral*—are sufficient in almost all practical cases. The influence of variations in abundance, gravity, and temperature are discussed (§§ 9–10). If the opacity depends wholly on the interaction of electrons and ions, the lines of any element will have a *maximum* (for varying temperature) only *if some other element of easier ionization is present*, and the *position of the maximum will depend on the relative abundance of the two* (§§ 13–16). Opacity due to interaction of neutral atoms and electrons produces similar effects (§ 17). Equations are given for both cases and for the combination of the two.

Applications.—With the aid of *empirical expressions for the surface gravity in stars of different temperatures* (main sequence and giants) [§ 18], the conditions for maxima of lines of *H*, Mg^+, and Fe^+ are worked out for atmospheres composed *mainly of hydrogen* (§§ 19–20) and of *easily ionized atoms* like sodium (§ 21). The *observed low temperature for* Fe^+ (class F5) and the *persistence of arc lines of Fe in the hotter stars* can be *explained in the former case*, but not in the latter. *Numerical values for the pressure* are derived from the writer's data for the solar atmosphere (§ 22). From the maxima of Fe^+ and Ti^+ it follows (§ 23) that *hydrogen is from 1000 to 2000 times more abundant than all the metals together*. For a *model atmosphere*, containing 1 atom of *K* to 3 of *Na*, 12 of *Fe*, and 16,000 of *H*, the *electron pressure* and *amount of matter above the photosphere* are calculated for main-sequence stars and giants (Figs. 1, 2; § 25). The *numbers of atoms engaged in producing given spectral lines* are then computed, approximate account being taken of the transition probabilities (§ 26; Figs. 3–8). The *computed temperatures of maximum agree closely with observation*, including the *previously anomalous* (K) *line of* Ca^+. A *moderate disagreement* for *H* and Mg^+ may be explained by a contribution of neutral atoms to opacity (Figs. 6–7), but more probably arises from *distortion of the energy-curve in stars of class A*, making their *color temperatures too high*. The outstanding discordance in the absolute-magnitude effect for *H* and *He* at high temperatures is attributed to the Stark effect. The still more anomalous strength of the Balmer series and the enhanced lines in red giants is ascribed tentatively to large differences in temperature within the atmosphere.

1. It is generally recognized that the intensity of lines in stellar spectra depends on the depth to which we can "see down" into the outer layers, and hence on the law of opacity.[3]

[1] *Contributions from the Mount Wilson Observatory, Carnegie Institution of Washington*, No. 477.

[2] Research Associate of the Mount Wilson Observatory, Carnegie Institution of Washington.

[3] E. A. Milne, *Monthly Notices of the Royal Astronomical Society*, **89**, 17 and 157, 1928; W. H. McCrea, *ibid.*, **91**, 836, 1931; E. A. Milne and S. Chandrasekhar, *ibid.*, **92**, 150,

The principal source of general opacity is the interaction of electrons and ions.[4] Part of this arises from "free-free" transitions which, according to Pannekoek, produce an atomic absorption coefficient for radiation of frequency ν

$$\alpha_0(\nu) = C_0 \frac{PFaZ^2}{T^{\frac{3}{2}}\nu^3}, \tag{1}$$

where

$$C_0 = \frac{4\sqrt{6\pi}}{9} \frac{e^6}{hc(mk)^{\frac{3}{2}}} = 2.653 \times 10^{24}, \tag{2}$$

m being the mass of the electron, P the electron pressure, Z the charge of the ion, a the fraction of the total mass which is formed by ions of this sort, and F a numerical factor. The equation has been derived only for "hydrogenic" atoms, but is probably a tolerable approximation for others—especially for the relatively small energy changes involved in absorption in the visible region.

The "bound-free" absorption due to photoelectric ionization of a given atom or ion increases abruptly just beyond the limit of each series in its spectrum. Smoothing these irregularities by an approximate method, Pannekoek derives an equation which may be written

$$\alpha_1(\nu) = C_0 \frac{PF'a}{T^{\frac{3}{2}}\nu^3} \frac{Z^2}{h\nu_1} \{(h\nu - kT)e^{h\nu/kT} + kT\}, \tag{3}$$

where F' is a new constant, $h\nu_1$ the energy of ionization, Z the charge, and a the abundance of atoms in which the ionization has occurred. Since the atom has here also been assumed to be "hydro-

1932. Some results of the present investigation were reported to the American Astronomical Society on December 30, 1931 (see *Publications of the American Astronomical Society*, **7**, 99, 1932).

[4] J. A. Gaunt, *Philosophical Transactions of the Royal Society of London*, A, **229**, 163, 1930; A. Pannekoek, *Monthly Notices of the Royal Astronomical Society*, **91**, 162, 1931; S. Chandrasekhar, *Proceedings of the Royal Society of London*, A, **135**, 472, 1932; B. Strömgren, *Zeitschrift für Astrophysik*, **4**, 119, 1932; A. S. Eddington, *Monthly Notices of the Royal Astronomical Society*, **92**, 364, 1932; R. C. Majumdar, *Astronomische Nachrichten*, **247**, 217, 1932.

genic," we may set $h\nu_1 = Z^2\chi_0$, where χ_0 is the energy of ionization of hydrogen.

To this approximation atoms of all degrees of ionization produce the same bound-free absorption. The reason is simple. Absorption of this type is produced only by atoms so highly excited that the energy χ_r required to complete their ionization is less than $h\nu$. The relative numbers of such atoms and of those in the next stage of ionization are determined by equations of the usual type, in which χ_r appears instead of the principal ionization potential. They are, therefore, independent of the latter. This formula is inapplicable beyond the principal series limit; but all wave-lengths of astrophysical importance are on the safe side. For a mixture, in comparable proportions, of many elements with overlapping limits, the approximation should be good. For hydrogen, Pannekoek's equations[5] give

$$a_1(\nu) = C_0 \frac{PF'a}{T^{\frac{3}{2}}\nu^3} \frac{\chi_0}{kT} \sum_m^\infty \frac{2}{n(n^2-\frac{1}{4})^2} e^{\chi_0/n^2kT} . \qquad (4)$$

Between λ 3646 and λ 8203 we must set $m = 3$. The ratio, S, of the sum to its first term at various temperatures is

$T =$	2900°	5800°	8700°	11,600°	23,200°	∞
$S =$	1.018	1.078	1.128	1.165	1.239	1.368 .

The abrupt changes at the series limits have been discussed by McCrea.[6]

2. It is convenient to express the energies in electron-volts, setting[7]

$$\left.\begin{aligned} I_0 &= \frac{300\chi_0}{e} = 13.54 , \\ V &= \frac{300h\nu}{e} = \frac{1.234 \times 10^{-4}}{\lambda} , \\ vT &= \frac{300kT}{e} = \frac{1}{11{,}600} T . \end{aligned}\right\} \qquad (5)$$

If A is the atomic weight and M_0 the mass corresponding to unit atomic weight,[8] we have for the mass coefficient of absorption

[5] *Op. cit.*, p. 163 (bottom of page). [6] *Ibid.*, p. 850, 1931.

[7] There ought to be a standard notation for the useful constant here called v.

[8] Birge's notation (*Physical Review Supplement*, 1, 62, 1929). Most writers call this M_H, but it is *not* the mass of the hydrogen atom.

$\kappa(r) = a(\nu)/AM_0$. Also, if ϵ is the fraction by number of all the atoms present which belong to the given element, x_1 the fraction of these which are singly ionized, and μ the mean atomic weight (excluding the free electrons) *in grams*, $a\mu = \epsilon x_1 AM_0$. If, then, we set

$$C_1 = \frac{C_0}{c^3 I_0} = 7.27 \times 10^{-9}\,, \tag{6}$$

we find easily for free electrons and singly ionized atoms

$$\kappa_0(\nu) = C_1 \frac{\epsilon x_1}{\mu} \frac{P\lambda^3}{T^{\frac{3}{2}}} FI_0\,, \tag{7}$$

and for ionization of neutral atoms

$$\kappa_1(\nu) = C_1 \frac{\epsilon x_1}{\mu} \frac{P\lambda^3}{T^{\frac{3}{2}}} F'\,G'(\lambda, T)\,, \tag{8}$$

where

$$G'(\lambda, T) = (V - vT)e^{V/vT} + vT\,. \tag{9}$$

The factors F' are well determined theoretically and, under ordinary conditions, lie between 0.9 and 1.0. We may set $F' = 1$ without serious error. The F's are larger and have not yet been accurately determined. Since, according to Y. Sugiura[9], $F = 6$ approximately, we may set $FI_0 = 80$.

The contributions of different atoms to the opacity are additive. Summing for all elements and degrees of ionization, we find

$$\kappa(\nu) = C_1 \mu^{-1} P\lambda^3 T^{-\frac{3}{2}} [G'(\lambda, T) \sum\epsilon \sum x_r + FI_0 \sum\epsilon \sum r^2 x_r]\,, \tag{10}$$

where the sums extend from $r = 1$ upward. The fractions of all the atoms which are neutral, singly ionized, etc., are

$$\bar{x}_0 = \sum\epsilon x_0 \qquad \bar{x}_1 = \sum\epsilon x_1 \ldots\,,$$

so that we have

$$\kappa(\nu) = C_1 \mu^{-1} \lambda^3 P T^{-\frac{3}{2}} \{G'(\lambda, T) \sum\bar{x}_r + FI_0 \sum r^2 \bar{x}_r\}\,.$$

[9] *Scientific Papers of the Institute of Physical and Chemical Research, Tokyo*, **17**, 89, 1931.

When second ionization is negligible, we may write

$$\kappa(\nu) = C_1 \mu^{-1} \lambda^3 P T^{-\frac{3}{2}} \bar{x}_1 \, G(\lambda, T) \, , \tag{11}$$

where

$$G(\lambda, T) = (V - vT) e^{V/vT} + vT + FI_0 \, .$$

Hydrogen is an exception. From (4) we find for the visible region

$$\kappa(\nu) = C_1 \mu^{-1} \lambda^3 P T^{-\frac{3}{2}} \epsilon x_1 \, H(T) \, , \tag{12}$$

where

$$H(T) = \frac{1.60 S}{vT} \, e^{1.50/vT} + FI_0 \, . \tag{13}$$

Typical values of G and H are given in Table I.

TABLE I

vT	T	$H(T)$	$G(\lambda,T)$			
			$V=$ 2.0 $\lambda=6168$	2.5 4934	3.0 4112	3.5 3515
0.25	2900°	2710	5310	4.9×10^4	4.5×10^5	3.9×10^6
0.50	5800	150	163	377	1088	3370
0.75	8700	98	99	130	204	373
1.00	11600	88	88	99	121	164
2.00	23200	82	82	84	87	91

3. The mean absorption for the whole flux of radiation follows a quite different law. The mean wave-length varies as T^{-1}; hence the integrated absorption varies as $PT^{-\frac{3}{2}}$ for the free-free transitions and nearly as $PT^{-\frac{11}{2}}$ for the bound-free. The Rosseland mean, which follows this law, is to be used in dealing with the general radiative equilibrium, but when discussing the optical depth for a given region of the spectrum, the monochromatic absorption should be employed.

The writer is unable to agree in this respect with Milne and Chandrasekhar, who use the mean absorption. The very different dependence on temperature produces important differences in the results, as will be seen below.

At low temperatures, when the ions become few, encounters be-

tween free electrons and neutral atoms may have to be considered. We may assume, by analogy,

$$\kappa_2(\nu) = C_2 \lambda^3 P T^{-\frac{3}{2}} \mu^{-1} \bar{x}_0 \,. \tag{14}$$

Whether C_2 is even roughly constant as the temperature and composition vary is unknown; but it is probably small in comparison with C_0. In the cases where this type of opacity is sensible, we may set $\bar{x}_0 = 1$, which is equivalent to adding a small part of the opacity due to the ions.

When ionization ceases at low temperatures, all these forms of opacity disappear; but the gas will not actually be perfectly transparent. Rayleigh scattering at least will remain and give a very small value of κ, independent of the density.

Continuous absorption of the type connected with band spectra may be more important at the lowest stellar temperatures. This should give κ proportional to the gas pressure, when compounds are just forming. In the absence of detailed knowledge, we may take the simple relation

$$\kappa_3(\nu) = C_3 \mu^{-1} \,, \tag{15}$$

where C_3 may vary with λ, and from atom to atom.

4. We have now to find the strength of spectral lines according to these opacity laws. Milne has shown[10] that a surprisingly good approximation may be obtained by treating a stellar atmosphere as if it were an isothermal layer of gas, above an opaque photosphere located at a level corresponding to the optical depth τ_0 for radiations near the spectral lines but outside them. The actual lines will be nearly equal to those which would be produced by an atmosphere transparent except for selective absorption, and of this depth.

In such an atmosphere the numbers of neutral and singly ionized atoms above the level at which the total pressure is p are given by the equations

$$\mu g N_0 = \epsilon \int_0^p x_0 dp \,, \tag{16}$$

$$\mu g N_1 = \epsilon \int_0^p x_1 dp \,. \tag{17}$$

[10] *Op. cit.*, p. 3. The greater the value of $\kappa\nu/\kappa$, the better this approximation.

Summing for all stages of ionization, we have

$$\mu g N = \epsilon p \tag{18}$$

for the whole number of atoms of the given kind. The optical depth to this level is given by

$$g\tau(\nu) = \int_0^p \kappa(\nu) dp \ . \tag{19}$$

The equation of ionization may be written

$$x_r P = x_{r-1} K_r \ , \tag{20}$$

$$K_r = \frac{CB_r}{B_{r-1}} T^{\frac{5}{2}} e^{-I_r/vT} \ . \tag{21}$$

The partition functions B are defined by

$$B = \sum q e^{-E/vT} \ ,$$

where q is the statistical weight and E the excitation potential, and the summation extends over all the energy states of the neutral (or the ionized) atom. The ratio B_1/B_0 often differs considerably from that of the weights of the lowest states.[11]

In thermodynamic equilibrium

$$C = \frac{2(2\pi m)^{\frac{3}{2}} k^{\frac{5}{2}}}{h^3} = 0.666 \ .$$

Pannekoek has shown[12] that in an atmosphere of temperature T' exposed to radiation from a photosphere of effective temperature T, equations (20) and (21) may still be used if the *photospheric* temperature is introduced and C is multiplied by $\frac{1}{2}(T'/T)^{\frac{3}{2}}$. Adopting $T' = 0.86T$, after Eddington,[13] we have $C = 0.267$. From (10), (16)–(19), and (21), we find

$$\tau_1(\nu) = C_1 \lambda^3 T^{-\frac{3}{2}} \{G'(\lambda, T) \sum K_r N_{r-1} + F I_0 \sum r^2 K_r N_{r-1}\} \ .$$

[11] Cf. Russell, *Mt. Wilson Contr.*, No. 447; *Astrophysical Journal*, **75**, 340, 1932.

[12] *Handbuch der Astrophysik*, **3**, Part I, 289, 1930.

[13] *The Internal Constitution of the Stars*, pp. 323, 335, 1926.

When second ionization is negligible, we have

$$\tau_1(\nu) = C_1 \lambda^3 T^{-\frac{3}{2}} G(\lambda, T) \sum K_1 N_0 . \tag{22}$$

From (14) and (15) we find

$$\mu g \tau_2(\nu) = C_2 \lambda^3 T^{-\frac{3}{2}} \int_0^p P dp , \tag{23}$$

$$\mu g \tau_3(\nu) = C_3 p = C_3 \mu g \sum N . \tag{24}$$

We have also

$$P = (p - P) \sum \epsilon \sum r x_r . \tag{25}$$

In an isothermal atmosphere the K''s are constant, and the N's are integrals of rational functions of p. When n elements are present, these integrals, even when only one ionization has to be considered, involve an irreducible function of degree $n-1$ in the denominator. It is easy to prove that the roots of this function are all real and lie in the intervals between the successive values of K for the various elements[14] so that the integrals lead to logarithmic functions; but the general solution is practically useless if $n > 2$.

5. The solution for $n = 2$ has been given by the writer[15] and by Milne.[16] It may easily be extended to cover the case when a third element is present, but not at all ionized.

If ϵ and ϵ' are the abundances of the two active elements X and X', we have

$$\frac{p}{P} = 1 + \frac{(P+K)(P+K')}{Q(P+R)} , \tag{26}$$

where

$$Q = \epsilon K + \epsilon' K' \qquad QR = (\epsilon + \epsilon') K K' .$$

[14] Two elements with equal K may be lumped together.

[15] *Mt. Wilson Contr.*, No. 383; *Astrophysical Journal*, **70**, 76, 1929.

[16] *Monthly Notices of the Royal Astronomical Society*, **92**, 166, 1932.

We then find

$$\mu g N = \epsilon p ,$$

$$\left.\begin{aligned}\mu g N_1 = \frac{K\epsilon P}{Q}\left\{1+\frac{P+K'}{P+R}\right\} - (1-\epsilon)K\log\left(1+\frac{P}{K}\right) \\ + \frac{\epsilon'}{\epsilon+\epsilon'}\frac{R^2}{K}\log\left(1+\frac{P}{R}\right),\end{aligned}\right\} \quad (27)$$

$$\left.\begin{aligned}\mu g N_0 = \frac{\epsilon P}{Q}\left\{\frac{P(P+K')}{P+R}+Q-K\right\} + (1-\epsilon)K\log\left(1+\frac{P}{K}\right) \\ - \frac{\epsilon'}{\epsilon+\epsilon'}\frac{R^2}{K}\log\left(1+\frac{P}{R}\right).\end{aligned}\right\} \quad (28)$$

The values of N_0', N_1' are obtained by interchanging primed and unprimed letters.

We now find, after a little reckoning,

$$\left.\begin{aligned}\mu g(KN_0+K'N_0') = P^2+P(Q-K-K'+R)+(1-\epsilon)K^2\log\left(1+\frac{P}{K}\right) \\ +(1-\epsilon')K'^2\log\left(1+\frac{P}{K'}\right)-R^2\log\left(1+\frac{P}{R}\right),\end{aligned}\right\} \quad (29)$$

and, by (22),

$$\tau_1 = C_1\lambda^3 T^{-\frac{3}{2}} G(\lambda, T)(KN_0+K'N_0') . \quad (30)$$

A direct integration gives

$$\left.\begin{aligned}\mu g\tau_2 = \frac{C_2\lambda^3 T^{-\frac{3}{2}}}{Q}\Big|\tfrac{2}{3}P^3+\tfrac{1}{2}P^2(Q+K+K'-R) \\ +R(K-R)(K'-R)\left\{\log\left(1+\frac{P}{R}\right)-\frac{P}{P+R}\right\}\Big] .\end{aligned}\right\} \quad (31)$$

The proportion of inert gas is $\epsilon_0 - 1 - \epsilon - \epsilon'$. Let us now suppose that a fourth element, X'', is present in so small a proportion, ϵ'', that

it makes no sensible contribution to the pressure. We then find easily

$$\mu g N'' = \epsilon'' p \,,$$

$$\left.\begin{aligned} \mu g N_1'' = \frac{\epsilon'' K''}{Q}\Bigg[& 2P + (Q+K+K'-R-2K'') \log\left(1+\frac{P}{K''}\right) \\ & + \frac{R(K-R)(K'-R)}{(K''-R)^2}\left\{\frac{P(K''-R)}{R(P+R)} - \log\left(1+\frac{P}{R}\right)\right. \\ & \left. + \log\left(1+\frac{P}{K''}\right)\right\}\Bigg] . \end{aligned}\right\} \tag{32}$$

When $K''=R$ this goes over into

$$\left.\begin{aligned} \mu g N_1'' = \frac{\epsilon'' K''}{Q}\Bigg\{ & 2P + (Q+K+K'-3R) \log\left(1+\frac{P}{R}\right) \\ & + (K-R)(K'-R)\,\frac{P(P+2R)}{2R(P+R)^2}\Bigg\} . \end{aligned}\right\} \tag{33}$$

6. When there is but one active element, X, we may set $\epsilon'=0$. Then $Q=\epsilon K$, $R=K'$, and

$$\mu g N = \frac{P^2}{K} + (1+\epsilon)P \,, \tag{34}$$

$$\mu g N_1 = 2P - (1-\epsilon)K \log\left(1+\frac{P}{K}\right) , \tag{35}$$

$$\mu g N_0 = \frac{P^2}{K} - (1-\epsilon)\left\{P - K \log\left(1+\frac{P}{K}\right)\right\} , \tag{36}$$

$$\tau_1 = C_1 \lambda^3 T^{-\frac{3}{2}} G(\lambda, T) K N_0 \,, \tag{37}$$

$$\mu g \tau_2 = \frac{C_2 \lambda^3 T^{-\frac{3}{2}}}{2\epsilon}\left\{\frac{4}{3}\frac{P^3}{K} + (1+\epsilon)P^2\right\} , \tag{38}$$

$$\mu g N_1'' = \frac{\epsilon'' K''}{\epsilon K}\left[2P + \{(1+\epsilon)K - 2K''\} \log\left(1+\frac{P}{K''}\right)\right] , \tag{39}$$

$$\mu g N_0'' = \frac{\epsilon'' K''}{\epsilon K}\left[\frac{P^2}{K''} + \{(1+\epsilon)K - 2K''\}\left\{\frac{P}{K''} - \log\left(1+\frac{P}{K''}\right)\right\}\right] . \tag{40}$$

Equations (34)–(36) have been given by Milne.[17]

[17] *Ibid.*, p. 169.

7. If X' is completely ionized, we may set $K'=\infty$, $R=[(\epsilon+\epsilon')/\epsilon']K$, and obtain (after expanding certain logarithms in series)

$$\mu g N = \frac{\epsilon}{\epsilon'}\frac{P}{P+R}\{(1+\epsilon')P+K+\epsilon' R\}\,, \tag{41}$$

$$\mu g N_1 = \frac{\epsilon}{\epsilon'}\frac{PK}{P+R} - (1-\epsilon)K\log\left(1+\frac{P}{K}\right)+R\log\left(1+\frac{P}{R}\right), \tag{42}$$

$$\left.\begin{aligned}\mu g N_0 = \frac{\epsilon}{\epsilon'}\frac{P}{P+R}\{(1+\epsilon')P+\epsilon' R\}+(1-\epsilon)K\log\left(1+\frac{P}{K}\right)\\ -R\log\left(1+\frac{P}{R}\right),\end{aligned}\right\} \tag{43}$$

$$\left.\begin{aligned}\mu g(KN_0+K'N_0') = (1+\epsilon')\left(\tfrac{1}{2}P^2+\frac{\epsilon}{\epsilon'}KP\right)\\ +(1-\epsilon)K^2\log\left(1+\frac{P}{K}\right)-R^2\log\left(1+\frac{P}{R}\right).\end{aligned}\right\} \tag{44}$$

τ is then given by (30). We have also

$$N_0''+N_1'' = \frac{\epsilon''}{\epsilon}N\,,$$

$$\left.\begin{aligned}\mu g N_1'' = \frac{\epsilon''K''}{\epsilon'}\Bigg[(1+\epsilon')\log\left(1+\frac{P}{K''}\right)+\frac{R(K-R)}{(K''-R)^2}\left\{\frac{P(K''-R)}{R(P+R)}\right.\\ \left.-\log\left(1+\frac{P}{R}\right)+\log\left(1+\frac{P}{K''}\right)\right\}\Bigg].\end{aligned}\right\} \tag{45}$$

If there is no inert gas present, $\epsilon+\epsilon'=1$. If then the completely ionized material is in great excess, our equations become

$$\left.\begin{aligned}\mu g N &= 2\epsilon P\,,\\ \mu g N_1 &= 2\epsilon K\log\left(1+\frac{P}{K}\right),\\ \mu g N_0 &= 2\epsilon\left\{P-K\log\left(1+\frac{P}{K}\right)\right\},\end{aligned}\right\} \tag{46}$$

which are those of Milne's problem III.[18]

[18] *Ibid.*, **89**, 26, 1928.

8. We may now test Milne's rule that the quantity $n = PN_1/2KN_0$ is not far from unity. He has proved that $n = 1$ exactly when only one element is present. The deviation will be greatest for an element present only in traces. In this case we may use equations (39) and (40), which assume the presence of two other elements, one partly ionized and one inert.

If we set $P/K = y$, $P/K'' = y''$, these may be written

$$\mu g N_1'' = \frac{\epsilon''}{\epsilon} 2P\{yf_2(y'') + (1+\epsilon)f_1 y''\}\,, \tag{47}$$

$$\mu g N_0'' = \frac{\epsilon''}{\epsilon}\frac{P^2}{K''}\{yf_3(y'') + (1+\epsilon)f_2(y'')\}\,, \tag{48}$$

when

$$f_1(y) = \frac{1}{2y}\log(1+y) = \tfrac{1}{2} - \tfrac{1}{4}y + \tfrac{1}{6}y^2 \ldots\,, \tag{49}$$

$$f_2(y) = \frac{1}{y} - \frac{1}{y^2}\log(1+y) = \tfrac{1}{2} - \tfrac{1}{3}y + \tfrac{1}{4}y^2 \ldots\,, \tag{50}$$

$$f_3(y) = \frac{1}{y} - \frac{2}{y^2} + \frac{2}{y^3}\log(1+y) = \tfrac{2}{3} - \tfrac{1}{2}y + \tfrac{2}{5}y^2 \ldots\,. \tag{51}$$

Note that

$$yf_2(y) = 1 - 2f_1(y)\,, \qquad yf_3(y) = 1 - 2f_2(y)\,. \tag{52}$$

These functions are given in Table II, in which $1/(2+y)$ is chosen as argument to make the tabular differences run smoothly. When y is large—that is, when the active element X is slightly ionized—$n = f_2(y'')/f_3(y'')$ and ranges only from $\frac{3}{4}$ to 1. When y is small, $n = f_1(y'')/f_2(y'')$. This becomes large when y'' is great. The error of Milne's approximation is serious only for an element which is just beginning to be ionized in the presence of an excess of highly ionized atoms. We have by (52)

$$n = \frac{1+\epsilon+\{2y-(1+\epsilon)y''\}f_2(y'')}{1+\epsilon+\{2y-(1+\epsilon)y''\}f_3(y'')}\,.$$

Milne's approximation is therefore exact if $2y = (1+\epsilon)y''$. Now by (21), $y''/y = B_1'' B_0/B_0'' B_1 e^{(I_1 - I_1')/\nu T}$. Hence this happens when X''

is slightly easier to ionize than X; but $I-I''$ will be only a fraction of a volt. For the active element X itself, $y''=y$ and

$$n=\frac{1-(1-\epsilon)f_1(y)}{1-(1-\epsilon)f_2(y)},$$

so that $n=1$ when y is large or small. It has a minimum, 0.93, when $\epsilon=0$, $y=3$.

TABLE II

VALUES OF $f(y)$

y	$1/(2+y)$	$f_1(y)$	$f_2(y)$	$f_3(y)$
∞	0.00	0.000	0.000	0.000
98	.01	.023	.010	.010
48	.02	.041	.019	.020
18	.05	.080	.046	.050
8	.10	.137	.091	.102
4.67	.15	.186	.135	.157
3	.20	.231	.179	.214
2	.25	.275	.225	.275
1.33	.30	.318	.273	.340
0.86	.35	.361	.324	.411
0.50	.40	.406	.378	.488
0.22	.45	.452	.437	.572
0.00	0.50	0.500	0.500	0.667

Equations (26)–(30) may now be simplified. If we set

$$\epsilon=a(1-\epsilon_0), \qquad \epsilon'=a'(1-\epsilon_0), \tag{53}$$

and $P/K=y$, $P/K'=y'$, $P/R=y_r$, we find

$$a+a'=1, \qquad y_r=a'y+ay', \tag{54}$$

$$\mu gN=aP\left(1-\epsilon_0+\frac{(1+y)(1+y')}{1+y_r}\right), \tag{55}$$

$$\mu g\tau_1=C_1\lambda^3T^{-\frac{3}{2}}G(\lambda, T)P^2\{1-(1-\epsilon)f_2(y)-(1-\epsilon')f_2(y')+f_2(y_r)\}, \tag{56}$$

$$\mu gN_1=P\left\{a\left(\frac{y'}{y_r}+\frac{1+y'}{1+y_r}\right)-2(1-\epsilon)f_1(y)+2\,\frac{a'y}{y_r}f_1(y_r)\right\}. \tag{57}$$

$$\mu g N_0 = \frac{P^2}{K}\left\{\frac{a(1+y')}{1+y_r} - (1-\epsilon)f_2(y) + a' f_2(y_r)\right\}, \tag{58}$$

$$\left.\begin{aligned}\mu g \tau_2 = \frac{C_2 \lambda^3 T^{-\frac{3}{2}} P^2}{1-\epsilon_0}\Bigg\{1 - \tfrac{1}{2}\epsilon_0 + \tfrac{2}{3}\frac{yy'}{y_r} \\ + \frac{(y_r - y)(y_r - y')}{y_r^2}\left(\frac{1}{1+y_r} - f_2(y_r) - \tfrac{1}{2}\right)\Bigg\}.\end{aligned}\right\} \tag{59}$$

9. Consider now an atmosphere in which practically all the opacity arises from the interaction of electrons and ions produced from a single active element, X, of abundance ϵ—the rest of the gas being inert. The number of atoms effective in producing a given line may be taken as that of the atoms, in the appropriate state, which lie above the level at which τ has some fixed value τ_0. Equations (35)–(37) may then be written

$$\mu g N_1 = 2P\{1 - (1-\epsilon)f_1(y)\}, \tag{60}$$

$$\mu g N_0 = \frac{P^2}{K}\{1 - (1-\epsilon)f_2(y)\}, \tag{61}$$

$$\mu g \tau_1 = C_1 \lambda^3 T^{-\frac{3}{2}} G(\lambda, T) P^2 \{1 - (1-\epsilon)f_2(y)\}. \tag{62}$$

The number M of atoms in any state, with excitation potential E and weight q, is given by

$$M = \frac{q}{B} N e^{-E/vT}. \tag{63}$$

Setting $\tau = \tau_0$, eliminating P, and using (21), we have

$$M_0 = \frac{\tau_0 q_1}{C_1 B_1 C} \lambda^{-3} T^{-1} e^{(I_1 - E_0)/vT} G(\lambda, T)^{-1}, \tag{64}$$

$$M_1 = 2(\mu g)^{-\frac{1}{2}} \frac{q_1}{B_1}\left(\frac{\tau_0}{C_1}\right)^{\frac{1}{2}} \lambda^{-\frac{3}{2}} T^{\frac{3}{4}} e^{-E_1/vT} G(\lambda, T)^{-\frac{1}{2}} f(y, \epsilon), \tag{65}$$

where

$$f(y, \epsilon) = \{1 - (1-\epsilon)f_1(y)\}\{1 - (1-\epsilon)f_2(y)\}^{-\frac{1}{2}}, \tag{66}$$

which always lies between 0.707 and 1. N_0 and N_1 are obtained by setting $E = 0$. For hydrogen we must substitute $H(T)$ for $G(\lambda, T)$.

From these equations it follows that:

a) The *abundance* of X has no effect on the intensities of its arc lines, and only an insignificant direct influence on its enhanced lines. The inert atoms, however, may have an indirect effect by altering the mean atomic weight μ. Dilution of a metallic atmosphere, too cool to ionize hydrogen or helium, with these gases strengthens the enhanced lines. The arc lines are unchanged. This effect approaches a limit as the dilution becomes great.

TABLE III

VALUES OF $\alpha(Vz)$

w	Vz							
	0	1	2	3	4	5	6	7
20	0.02	0.08	0.22	0.46	0.72	0.90	0.98	1.00
30	.02	.06	.15	.35	.62	.84	.95	0.99
40	.01	.04	.12	.29	.55	.79	.92	0.98
50	0.01	0.03	0.10	0.24	0.49	0.74	0.90	0.97

b) The *surface gravity* has no effect on the arc lines of X. The enhanced lines show a positive effect (stronger in giants) of normal amount (varying as $g^{-\frac{1}{2}}$) and the same at all temperatures (barring higher ionization).[19]

c) The *temperature* influences several factors. If $z=1/vT$ and $w=FI_0/V$, we have

$$G(\lambda, T)=(V-z^{-1})e^{Vz}+z^{-1}+FI_0\,,$$

$$\frac{d\log G(\lambda, T)}{Vdz}=\frac{V^2z^2-Vz+1-e^{-Vz}}{V^2z^2-Vz+(Vz+wV^2z^2)e^{-Vz}}=\alpha(Vz)=\alpha\,\frac{V}{vT}\,. \quad (67)$$

When $z=0$, $\alpha=1/2w$; when $z=\infty$, $\alpha=1$. For the observable spectrum, V lies between 1.6 and 4. With $FI=80$ we have to deal with values of w ranging from 20 to 50. The corresponding values of α are given in Table III. We have then, by (64),

$$\frac{d}{dz}\log M_0=vT-\alpha V+I-E_0\,. \quad (68)$$

[19] Cf. Milne and Chandrasekhar, *op. cit.*, p. 170.

Hence M_0 decreases as T increases, unless $E_0+aV>I_1+vT$. Now the excitation potential for the upper state of any line is $E+V$, since V measures the energy drop in the line itself. This value is less than I_1 except for a few states in which two electrons are excited. All arc lines (except the very few having such upper states) will therefore steadily increase in intensity with falling temperature, until the ionization becomes so small that other sources of opacity have to be considered.

For hydrogen we set $d \log H(T)/dz=\beta(z)$ and find (determining $d \log S/dz$ graphically)

$z=\frac{1}{vT}$	0	0.5	1	2	3	4	6	8	10
β.....	0.02	0.08	0.22	0.90	1.53	1.68	1.67	1.63	1.60
β'.....	0.04	1.61	3.58	3.86	3.71	3.63	3.54	3.50	3.48

(The values β' apply beyond the limit of the Balmer series.) We have then

$$\frac{d \log M_0}{dz}=vT+I-E_0-\beta\,. \tag{69}$$

For cases in which most of the electrons come from hydrogen, the lines of the Balmer series should therefore decrease steadily with rising temperature.

Milne finds a minimum for these lines because his formula for K contains $T^{-\frac{9}{2}}$ instead of $T^{-\frac{3}{2}}$ (at lower temperatures, $T^{-\frac{11}{2}}$). This substitutes T^2 (or T^3) for T^{-1} in the expression for M_0, which then still increases at high temperatures. The weakening of the hydrogen lines in classes B and O can be explained on this theory only if there is a large increase of opacity due to the ionization of helium and higher ionizations of other elements, a condition which demands that these elements shall be more abundant than the hydrogen.

For the enhanced lines, we have from (60) and (49)

$$\frac{d \log N_1}{dz}=\frac{d \log P}{dz}+\frac{(1-a)\left\{f_1(y)-\frac{1}{2+2y}\right\}}{1-(1-\epsilon)f_1(y)}\,\frac{d \log y}{dz}\,,$$

and from (62) and (50)

$$\tfrac{3}{2}vT+aV+2\frac{d\log P}{dz}+\frac{2(1-\epsilon)\left\{f_2(y)-\dfrac{1}{2+2y}\right\}}{1-(1-\epsilon)f_2(y)}\frac{d\log y}{dz}=0\,,$$

also

$$\frac{d\log y}{dz}=\frac{d\log P}{dz}-\frac{d\log K}{dz}=\frac{d\log P}{dz}+I+\tfrac{5}{2}vT\,.$$

We then find

$$\frac{d\log N_1}{dz}=-\tfrac{3}{4}vT-\tfrac{1}{2}aV+\frac{(1-\epsilon)\{f_1(y)-f_2(y)\}}{1-(1-\epsilon)f_1(y)}\,(I+\tfrac{7}{2}vT-\tfrac{1}{2}aV)\,,$$

and may write

$$\frac{d\log M_1}{dz}=-E_1-\tfrac{3}{4}vT-\tfrac{1}{2}aV+\gamma(y,\epsilon)(I+\tfrac{7}{2}vT-\tfrac{1}{2}aV)\,. \qquad (70)$$

The factor γ is always small. It diminishes the rate of increase of the enhanced lines with rising temperature, but will not reverse it for any known lines (see §15).

TABLE IV

VALUES OF $\gamma(y,\epsilon)$

ϵ	$1/(2+y)$					
	0.00	0.10	0.20	0.30	0.40	0.50
0	0.000	0.053	0.068	0.066	0.047	.000
$\frac{1}{2}$	0.000	0.025	0.030	0.027	0.017	.000

Second ionization, of course, causes all enhanced lines to have maxima.

10. Consider next a constituent X'' which is present in such

amount that it contributes practically nothing to the electron pressure. From (47), (48), (60), (61), we have

$$\frac{N_0''}{N_0}=\frac{\epsilon''}{\epsilon}\,\frac{K}{K''}\,\frac{yf_3(y'')+(1+\epsilon)f_2(y'')}{1-(1-\epsilon)f_2(y)}\,, \tag{71}$$

$$\frac{N_1''}{N_1}=\frac{\epsilon''}{\epsilon}\,\frac{yf_2(y'')+(1+\epsilon)f_1(y'')}{1-(1-\epsilon)f_1(y)}\,. \tag{72}$$

The intensity of the lines of X'' depends on its abundance, ϵ''/ϵ, relative to the opacity-producing element X. Call this a''. We may now distinguish four typical cases.

Case 1.—X and X'' both slightly ionized. Using (52) and $Ky=K''y''$, we find

$$N_0''=N_0a'', \qquad N_1''=\frac{N_1a''K''}{K}=N_1a''e^{(I-I'')z}\,. \tag{73}$$

The arc lines are unaffected by gravity; the enhanced lines show a normal positive influence. For the temperature effect we find

$$\frac{d}{dz}\log M_0''=vT+I-\alpha V-E_0''\,, \tag{74}$$

$$\frac{d}{dz}\log M_1''=-\tfrac{3}{4}vT-\tfrac{1}{2}\alpha V+I-I''-E_1''\,, \tag{75}$$

the term in γ vanishing, for both small and large y. The arc lines strengthen with falling temperature, unless $E_0''>I+vT-\alpha V$.

If I'' is considerably less than I, the ultimate enhanced lines grow stronger as the temperature falls, and so may subordinate lines for which E_1'' is small. This increase has a limit. When the temperature falls so low that N_1'' becomes comparable with N_1, X'' takes over the rôle of active element. At lower temperatures its enhanced lines weaken.

Case 2.—X slightly, X'' heavily ionized. We have

$$N_0''=\tfrac{2}{3}N_0a''\,\frac{K}{K''}\,y\,, \qquad N_1''=\tfrac{1}{2}N_1a''y\,.$$

But by (60), (61), $y = P/K = 2N_0/N_1$. Hence

$$N_0'' = \tfrac{4}{3}a'' \frac{K}{K''} \frac{N_0^2}{N_1}, \qquad N_1'' = a'' N_0 . \tag{76}$$

The enhanced lines of X'' now behave in all respects like the arc lines of X. They are insensitive to gravity and weaken rapidly with rising temperature on account of the increasing opacity due to the ionization of X. For the arc lines $N_0 \propto (\mu g)^{\frac{1}{2}}$. These lines show a negative absolute-magnitude effect. Their decrease with rising temperature is unusually rapid.

Case 3.—X heavily, X'' slightly ionized.

$$N_0'' = \frac{2a'' N_0 K}{K'' y''} = a'' N_1 , \tag{77}$$

$$N_1'' = \frac{a'' N_1}{y''} \log (1+y'') = \frac{a''}{2} \frac{K''}{K} \frac{N_1^2}{N_0} \log (1+y'') ,$$

$$= \frac{(1+\epsilon) a'' K''}{\mu g} \log (1+y'') , \tag{78}$$

by (60), (61). Here the arc lines behave as enhanced lines usually do with regard to both temperature and gravity. (This happens for the Balmer series in the cooler stars, where only the metals are ionized.) For the enhanced lines the absolute-magnitude effect is twice as great as usual, and the increase with temperature is rapid. The optical depth τ_0 has disappeared from (78), although it enters indirectly, since $y'' \propto \tau_0^{\frac{1}{2}}$. Under the assumed conditions, X'' is considerably ionized only at high levels, and a considerable increase in τ_0 takes in many more neutral atoms, but few additional ions.

Case 4.—X and X'' both heavily ionized.

$$N_0'' = \frac{N_0 a'' K}{K''} , \qquad N_1'' = N_1 a'' . \tag{79}$$

The arc lines are again insensitive to gravity and the enhanced lines normally sensitive. The temperature effects are the same as if X'' were the active element.

Equations (71) and (72) show that the transitions between one and another of these cases will be smooth. They suffice for exact calculation in any desired case.

Equations (73), (76), (77), (78), and (79) depend only on the assumption of an isothermal atmosphere and are true for all laws of opacity. The dependence on gravity and temperature will differ with the assumed law; but the reversed sensitiveness to gravity for arc lines in case 2 and the increased effect for enhanced lines in case 3 are general properties as long as the opacity depends on ionization. These effects have been previously described by Milne,[20] who has shown that the relations are qualitatively true, though to a less degree, when X'' and X are of comparable abundance.

11. Second and higher ionizations remain to be considered. Under stellar conditions the second ionization of any element is insensible until the first is practically complete, and, when it happens, there is always a large excess of electrons arising from other sources. Equations (77)–(79) may therefore be applied in all cases, substituting I_2'' for I'', N_1'' for N_0'', and N_2'' for N_1''. We may distinguish three groups of elements:

i) $I''<13.6$. Ca^+, Sr^+, Ba^+, and, to a less degree, Sc^+, Y^+, La^+.—Second ionization precedes that of hydrogen but begins when the first ionization of the metals is practically complete. At this stage we have case 3. Ultimate lines of X''^+ are normally sensitive to gravity and almost insensitive to temperature. Those of X''^{++} (if observable) would show exceptional sensitiveness to gravity. If the ionization of X^+ is far advanced before that of hydrogen begins, we have case 4. The lines of X''^+ are now insensitive to gravity, and decrease with rising temperature. When hydrogen begins to contribute more electrons than the metals, we pass to case 2. The lines of X''^+ are stronger in dwarfs than in giants and decrease very rapidly with rising temperature. Finally, when most of the hydrogen is ionized, we return to case 4.

ii) $I''=13.6$ (approximately). Ti^+, Zr^+, and the rare-earths.—Lines of X''^+ behave as hydrogen lines of the same excitation potential would do. Starting with case 3, before the hydrogen is ionized,

[20] *Monthly Notices of the Royal Astronomical Society*, **89**, 166, 1928 (case 3); *ibid.*, **92**, 161, 1931 (case 2).

they fade rapidly as ionization goes on and the opacity increases, and become insensitive to gravity.

iii) $I''>13.6$. Mg^+, Sc^+, Fe^+, the light-elements, and all higher ionizations.—Second ionization begins with case 3 and goes over into case 4. Ultimate lines of X''^+ at their maximum intensity are normally sensitive to gravity, but lose this as they fade out; subordinate lines lose most of the gravity effect before they reach their maxima. Ultimate lines of X''^{++} would be exceptionally sensitive to gravity at their first appearance. By the time that the highly excited subordinate lines (which alone are observable) appear, this sensitiveness drops to about normal. When lines of X''^+ and X''^{++} are both visible, they will all be stronger in giants than in dwarfs, but the effect will be more pronounced for the higher ionization.

12. The observable lines of neutral helium will behave in the same way, being sensitive to gravity at temperatures below their maxima, and insensitive as they fade out in the hottest stars (cf. α Cygni!).

For the Balmer lines we have $I=13.5$, $E=10.1$. If the atmosphere were composed entirely of hydrogen, they would increase steadily, down to the lowest temperatures; (69) gives

$$\frac{d}{dz}\log M = 3.4+vT-\beta = +5 \text{ (for B stars).}$$

With metals present, this increase will cease as soon as they contribute about half the electrons. When z has increased by 0.2 (or T dropped by some 1500°), the hydrogen will be almost all neutral, we shall have case 3, and by (70) and (77), neglecting γ,

$$\frac{d}{dz}\log M = -\tfrac{3}{4}vT-\tfrac{1}{2}\alpha V-10.1 = -10 \text{ (classes F and G).}$$

When the temperature has fallen so low that most of the metallic atoms are neutral, we have case 1 and, by (74),

$$\frac{d}{dz}\log M = vT-\alpha V-10.1+I = -6\,,$$

setting $I=6$.

The intensity of the Balmer lines will therefore have a rather

sharp maximum near the point where ionized hydrogen and ionized metallic atoms are equally numerous, from which it will diminish slowly with rising temperature, and much more rapidly with falling. At low temperatures the rate of diminution (for equal changes in $1/T$, or the color index) will fall off. The lines will be sensitive to gravity in the region where the intensity falls most steeply, but not elsewhere.

The presence of several metallic constituents will smear out the transition between the second and third portions of the curve without altering its general shape.

13. The existence of a maximum intensity for any stellar line depends essentially on the presence of *other* sources of opacity than the ionization of the atoms or ions which produce it. This circumstance is implied by Milne's equations, but deserves emphasis on account of its importance.

The general problem of a maximum, where two or more elements are being ionized, is complex; but the case in which one element X is partly ionized, in the presence of any amount of material which is completely or insensibly ionized, is simple. In this case $y'=0$ in (54) and (59), and we have

$$\mu g N = aKy\left\{\frac{ay}{1+a'y}+2-\epsilon_0\right\}=aKy\,u\,, \tag{80}$$

$$\mu g N_1 = Ky\left\{\frac{a}{1+a'y}+2f_1(a'y)-2(a'+a\epsilon_0)f_1(y)\right\}=Ky\,u_1\,, \tag{81}$$

$$\mu g N_0 = Ky^2\left\{\frac{a}{1+a'y}+a'f_2(a'y)-(a'+a\epsilon_0)f_2(y)\right\}=Ky^2u_0\,, \tag{82}$$

$$\frac{\mu g\tau_1}{C_1\lambda^3 T^{-\frac{2}{3}}G(\lambda,\,T)}=K^2y^2\{\tfrac{1}{2}(1+a'-a'\epsilon_0)+f_2(a'y)-(a'+a\epsilon_0)f_2(y)\}=K^2y^2u_2\,, \tag{83}$$

$$\frac{a'(1-\epsilon_0)\mu g\tau_2}{C_2\lambda^3 T^{-\frac{3}{2}}}=K^2y^2\left\{\tfrac{1}{2}(1+a'-a'\epsilon_0)+af_2(a'y)-\frac{a}{1+a'y}\right\}=K^2y^2u_3\,. \tag{84}$$

Note that

$$u_2=u_0+u_3\,. \tag{85}$$

We may take y and z as independent variables. By (49) and (50)

$$d\{yf_1(a'y)\} = \frac{dy}{2(1+a'y)}\ ; \qquad d\{y^2f_2(a'y)\} = \frac{y\,dy}{1+a'y}\ ,$$

whence

$$d(yu) = (1+y)Y\,dy\ , \qquad d(yu_1) = a\ Y\,dy\ ,$$

$$d(y^2u_0) = ay\ Y\,dy\ , \qquad d(y^2u_2) = y(1+a'y)\ Y\,dy\ ,$$

$$d(y^2u_3) = a'y(1+y)\ Y\,dy\ ,$$

where

$$(1+y)(1+a'y)^2Y = 1+2y+a'y^2+(1-\epsilon_0)(1+a'y)^2\ .$$

We now have

$$d\log M_0 = -(I+E+\tfrac{5}{2}vT)dz+\frac{d(y^2u_0)}{y^2u_0}-d\log g\ , \tag{86}$$

$$d\log \tau_1 = -(2I+\tfrac{7}{2}vT-\mathfrak{a}V)dz+\frac{d(y^2u_2)}{y^2u_2}-d\log g = 0\ , \tag{87}$$

whence

$$d\log M_0 = \{U_2(2I+\tfrac{7}{2}vT-\mathfrak{a}V)-(I+E+\tfrac{5}{2}vT)\}dz-(1-U_2)d\log g\ , \tag{88}$$

where

$$U_2(y,\ a',\ \epsilon_0) = \frac{au_2}{(1+a'y)u_0}\ .$$

The numerator and denominator are linear functions of ϵ_0 and the latter is finite; hence the values of U_2 for ϵ between 0 and 1 are intermediate between its values at these limits.

If a' becomes very small, while $a'y$ remains finite, y is great and $f_2(y)$ negligible, so that $u_0 = 1/(1+a'y)$, $u_2 = \frac{1}{2}+f_2(a'y)$, and $U_2(y,0,\epsilon_0) = \frac{1}{2}+f_2(a'y)$. If a' is small and y finite, $U_2 = 1$. When a is small, we have

$$f_2(y)-a'^2f_2(a'y') = \frac{a}{y}-\frac{1}{y^2}\log\left(\frac{1+ay}{1+a'y}\right) = \frac{a}{1+a'y}+\ \cdots\ ,$$

$$u_0 = a(2-\epsilon_0)f_2(y)\ , \qquad u_1 = 2a(2-\epsilon_0)f_1(y)\ , \qquad u_2 = 1-\tfrac{1}{2}\epsilon_0\ , \tag{89}$$

$$U_2(y,\ 1,\ \epsilon_0) = \frac{1}{2(1+y)f_2(y)}\ .$$

Values of U_2 are given in Table V. These quantities change remarkably little with ϵ_0 and a'; they depend almost entirely on $a'y$, that is, on the relative number of electrons liberated by the active and the

TABLE V

FOR MAXIMA OF SUBORDINATE LINES

$1/(2+a'y)$	$U_2(y, \epsilon_0, 0)$	$U_2(y, \frac{1}{2}, 0)$	$U_2(y, \frac{1}{2}, 1)$	$U_2(y, \epsilon_0, 1)$
0.00	0.500	0.500	0.500	0.500
.02	.519	.526	.530	.533
.05	.546	.557	.563	.566
.10	.591	.602	.604	.613
.15	.635	.645	.647	.656
.20	.679	.690	.695	.698
.25	.725	.733	.739	.740
.30	.774	.779	.785	.783
.35	.824	.827	.830	.830
.40	.878	.881	.884	.882
.45	0.936	0.936	0.937	0.937
0.50	1.000	1.000	1.000	1.000

wholly ionized element. This ratio is $N_1/N_1' = u_1/a'u$. When a' is small,

$$\frac{N_1}{N_1'} = \frac{1+2(1+a'y)f_1(a'y)}{a'y} . \tag{90}$$

As this ratio increases, the sensitiveness to gravity diminishes, some corresponding values being

N_1							
N_1'	0.00	0.22	0.43	1.05	1.86	4.4	∞
$1-U_2$	0.50	0.45	0.41	0.32	0.23	0.12	0.00

When $a'y$ is large, $d \log M/dz$ is negative; when $a'y$ is small, $d \log M/dz$ is positive, unless $E+aV > I+vT$. This condition is satisfied only when the atom in the upper state contains more than enough energy to ionize it. All other lines will have maxima when

$$U_2(y, a', \epsilon_0) = \frac{I+E+\frac{3}{2}vT}{2I+\frac{3}{2}vT-aV} . \tag{91}$$

If E is small, these arc lines will still show a pronounced absolute-magnitude effect at and beyond their maxima; if E is large, the effect will be small.

14. For the Balmer series, we may take $V=2.8$ (near $H\gamma$), $T=10{,}000°$, whence $vT=0.862$, $aV=1.18$, $U_2=0.894$. From Table V, setting $\epsilon_0=a'=0$ (since both are known to be small), we find $1/(2+a'y)=0.414$, $a'y=0.42$, and, by (90),

$$\frac{N_1}{N_1'}=\frac{2.18}{a'y}=5.2\ ;$$

the hydrogen, therefore, produces five times as many electrons as the metals.

The assumed temperature is rather high. If $T=7000°$, $U_2=0.932$, $a'y=0.245$, $N_1/N_1'=8.6$.

Since hydrogen furnishes most of the ions, it is better to substitute β for aV in (91). This gives $T=10{,}000°$, $7000°$; $a'y=0.56$, 0.48; $N_1/N_1'=4.0$, 4.6. There is strong evidence that in stellar atmospheres a' is 0.01 or less. The maximum of the Balmer lines, therefore, comes when the hydrogen is not more than 5 per cent ionized.

For any other element we have

$$\frac{y''}{y}=\frac{B_0}{2B_1e^{(I''-I)z}}\quad\left(\text{since }\frac{q_0}{q_1}=2\text{ for }H\right)$$

and, approximately, $N_1''/N_0''=2/y''$. For iron, $I''=7.83$ and $B_0/B_1=0.75$ (at $7000°$). With the Balmer lines at maximum at $7000°$, $N_1''/N_0''=15{,}000\ a'$, while at $10{,}000°$ it is $1000\ a'$. Hence, unless a' is extremely small, iron and all other common metals will be very highly ionized, and our basic assumption is verified.

For the higher ionizations in B stars, hydrogen is in great excess, and we may take $\epsilon_0=0$, $a'=1$, and neglect β, which is small. We now have $N_1/N_0=2f_1(y)/yf_2(y)$, and find by computation the accompanying tabulation. For maxima of observable lines the ionization is always high. This is Milne's problem III (modified by the change in the opacity formula).

	I	E	vT	U_2	y	N_1/N_0
He λ 4471	24.5	20.9	1.5	0.906	0.37	5.7
He^+ 4686	54.2	48.2	2.5	0.927	0.27	7.5

15. For enhanced lines of X we have

$$d \log M_1 = -(I+E+\tfrac{5}{2}vT)dz + \frac{d(yu_1)}{yu_1} - d \log g .$$

The condition for a maximum is

$$\frac{au_2}{(1+a'y)u_1} = U_1(y, a, \epsilon_0) = \tfrac{1}{2} + \frac{E+\tfrac{3}{4}vT+\tfrac{1}{2}aV}{2I+\tfrac{5}{2}vT-aV} . \qquad (92)$$

When a is small, we find, from (89), $1/U_1 = 4(1+y)f_1(y)$; U_1 ranges from $\frac{1}{2}$ to 0 as y goes from 0 to ∞, and there is no real solution. When a' is small, we have

$$U_1 = \frac{\frac{1}{2}+f_2(a'y)-\epsilon_0 f_2(y)}{1+2(1+a'y)\{f_1(a'y)-\epsilon_0 f_1(y)\}} .$$

Now $f_1(y) > f_2(y)$. If $a'y$ is finite, $f_1(y)$ and $f_2(y)$ are negligible and $U_1 < \frac{1}{2}$, there is no solution. But if y is finite, $a'y = 0$ and $U_1 = [1-\epsilon_0 f_2(y)]/[2-2\epsilon_0 f_1(y)]$. The maximum value is 0.534 when $\epsilon_0 = 0$, $y = 2$. There will be two real solutions, giving a maximum and a minimum close together, if $E_1 + 0.63vT + 0.53aV < 0.034I$; but this condition is not met by any known enhanced line.

Since $1-U_1$ is usually greater than 0.5, these lines are unusually sensitive to gravity.

16. The lines of other elements may have maxima and minima, owing to the change in opacity, produced by the ionization of X, even though they are ionized completely or not at all. In the first case N_0'', and in the second case N_1'', is proportional to N, the "amount of matter above the photosphere." If M'' is the number of atoms of either sort which are excited with potential E'', we have from (80)

$$d \log M'' = -(I+E''+\tfrac{5}{2}vT)dz + \frac{d(yu)}{yu} - d \log g . \qquad (93)$$

Eliminating y by (87), we find

$$d \log M = \{U_2''(2I+\tfrac{7}{2}vT-aV) - (I+E''+\tfrac{5}{2}vT)\}dz - (1-U_2'')dg, \quad (94)$$

when

$$U_2''(y, a', \epsilon_0) = \frac{(1+y)u_2}{(1+a'y)u} .$$

Here again U_2'' varies monotonically with ϵ_0, and

$$U_2''(y, a', 1) = \tfrac{1}{2} + f_2(a'y) - f_2(y) , \tag{95}$$

$$U_2''(y, a', 0) = \frac{(1+y)\{1 + a' + 2f_2(a'y) - 2a'f_2(y)\}}{4 + 2y + 2a'y} . \tag{96}$$

The first of these quantities is given in Table VI, and also the difference of the two. This difference is small, and the presence of inert gas again produces very little effect.

For a stationary value we have

$$U_2'' = \frac{I + E'' + \frac{5}{2}vT}{2I + \frac{7}{2}vT - aV} . \tag{97}$$

If this has any solution, it has two. The larger value of y corresponds to a maximum of M'', the smaller to a minimum. Between the two, the increasing opacity due to the ionization of X "raises the photosphere" and diminishes M''. If a' is large, there will be no such effect, even for ultimate lines; but if a' is small, it will appear even for considerable values of E'', while for ultimate lines the maximum and minimum will be widely separated.

When a' is small, we have, approximately,

$$U_2'' = \frac{(1+y)\{\frac{1}{2} + f_2(a'y) - \epsilon_0 f_2(y)\}}{y + (2 - \epsilon_0)(1 + a'y)} .$$

If y is finite, this becomes

$$U_2'' = \frac{(y+1)\{1 - \epsilon_0 f_2(y)\}}{y + 2 - \epsilon_0} ; \tag{98}$$

but if $a'y$ is finite, we have

$$U_2'' = \tfrac{1}{2} + f_2(a'y) . \tag{99}$$

Each equation gives an approximate value of one of the roots. In the interval in which y is large and $a'y$ small, U_2'' is very nearly unity; hence practically all the lines show maxima and minima.

TABLE VI

$1/(y+2)$	$U_2''(y, a', 1)-\frac{1}{2}$									$U_2''(y, a', 1)-U_2''(y, a', 0)$			
	$a'=0.75$	0.50	0.30	0.20	0.10	0.05	0.02	0.01	0.00	0.50	0.10	0.05	0.00
0.00.....	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.000	0.000	0.000	0.000
.02.....	.506	.517	.537	.560	.613	.685	.793	.863	.981	.003	.005	.004	.001
.05.....	.513	.536	.575	.614	.692	.773	.860	.900	.945	.006	.009	.008	.004
.10.....	.522	.558	.614	.661	.742	.806	.861	.883	.909	.009	.013	.014	.009
.15.....	.528	.573	.633	.680	.750	.798	.835	.850	.865	.012	.018	.021	.015
.20.....	.533	.580	.640	.682	.739	.776	.802	.811	.821	.014	.022	.024	.021
.30.....	.533	.576	.623	.652	.686	.705	.716	.722	.726	.014	.024	.027	.026
.40.....	.525	.552	.576	.590	.605	.619	.621	.622	.622	.012	.020	.024	.022
0.50.....	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.000	0.000	0.000	0.000
Max.*....	0.535	0.582	0.640	0.689	0.750	0.806	0.866	0.900	1.000	0.567	0.734	0.792	1.000

* The last line gives the maximum values of $U_2''(y, a', 1)$ and $U_2''(y, a', 0)$.

17. If the opacity arises from interaction of electrons and neutral atoms, we have

$$d \log \tau_2 = -(2I + \tfrac{7}{2}vT)dz + \frac{d(y^2 u_3)}{y^2 u_3} - d \log g = 0 , \qquad (100)$$

$$d \log M_0 = \{U_3(2I + \tfrac{7}{2}vT) - (I + E + \tfrac{5}{2}vT)\}dz - (1 - U_3)d \log g , \qquad (101)$$

where

$$U_3 = \frac{au_3}{a'(1+y)u_0} = \frac{(1+a'y)U_2 - a}{a'(1+y)} . \qquad (102)$$

When $y = 0$, $U_3 = 1$, and when $y = \infty$, $U_3 = \frac{1}{2}$, so that there are maxima for practically all arc lines, as before. When a' is very small, U_3 ranges from $\frac{1}{2}$ to $\frac{2}{3}$ as $a'y$ decreases from large to small values, and then from $\frac{2}{3}$ to 1 as y does the same. If $a' = 0$, the first part of this range is lost, and there is a maximum only if $E > \frac{1}{3}I - \frac{1}{6}vT$. Otherwise, M increases steadily with increasing z or falling temperature.

For an element of unchanged ionization we have, by (93) and (87),

$$d \log M'' = \{U_3''(2I + \tfrac{7}{2}vT) - (I + E'' + \tfrac{5}{2}vT)\}dz - (1 - U_3'')d \log g , \qquad (103)$$

where

$$U_3'' = \frac{u_3}{a'u} . \qquad (104)$$

When y is small,

$$U_3'' = \tfrac{1}{2} + \frac{ay'}{12 - 6\epsilon_0} ;$$

and when y is large,

$$U_3'' = \tfrac{1}{2} + \frac{a}{2a'(1 + a' - a\epsilon_0)y} .$$

As for the other law of opacity, there will be maxima and minima, provided that E does not exceed a certain limit, increasing with a'. When a' is small, we have

$$U_3'' = \frac{\{1 + a'y\}\{\tfrac{1}{2} + f_2(a'y)\} - 1}{a'y} = U_3 . \qquad (105)$$

As $a'y$ decreases from large to small values, U_3'' goes from $\frac{1}{2}$ to $\frac{2}{3}$; as y decreases to small values, U_3'' returns to $\frac{1}{2}$. Hence, if $E<\frac{1}{3}I-\frac{1}{6}vT$, there is at higher temperature a maximum in the first range and a minimum in the second; but if E exceeds this limit, M increases with T (or as z decreases) until opacity due to the ions becomes important.

When both types of opacity are sensible, as may occur in a slightly ionized hydrogen atmosphere, we have

$$\mu g(\tau_1+\tau_2)=C_1\lambda^3T^{-\frac{3}{2}}K^2y^2(Hu_2+Au_3)\,, \tag{106}$$

where $A=C_2/C_1a'(1-\epsilon_0)$.

Since $dH/dz=\beta$, we have

$$\left.\begin{aligned} d\log(\tau_1+\tau_2)=&-(2I+\tfrac{7}{2}vT-x\beta)\,dz\\ &+\frac{H(1+a'y)+Aa'(1+y)}{Hu_2+Au_3}\,\frac{Ydy}{y}-d\log g\,, \end{aligned}\right\} \tag{107}$$

where $x=Hu_2/(Hu_2+Au_3)$. Then

$$d\log M_0=U_{23}(2I+\tfrac{7}{2}vT-x\beta)-(I+E+\tfrac{5}{2}vT)dz-(1-U_{23})d\log g\,, \tag{108}$$

if

$$U_{23}=\frac{H(1+a'y)U_2+Aa'(1+y)U_3}{H(1+a'y)+Aa'(1+y)}\,.$$

If

$$\frac{\xi}{1-\xi}=\frac{H(1+a'y)}{Aa'(1+y)}\,, \tag{109}$$

$$U_{23}=\xi U_2+(1-\xi)U_3\,,\qquad x=\frac{\xi U_2}{U_{23}}\,. \tag{110}$$

For X'' we may use (108), substituting for U_{23}'',

$$U_{23}''=\xi U_2''+(1-\xi)U_3''=\frac{(1+y)u_0}{au}\,U_{23}\,. \tag{111}$$

From (12) and (14), $Aa'/H=\kappa_2/(1-\epsilon_0)\kappa_1$, where κ_1 and κ_2 measure the opacities due to equal numbers of ions and neutral atoms. Set

$$b=\frac{\kappa_2}{\kappa_1}=\frac{C_2}{C_1H(T)}\,. \tag{112}$$

This ratio varies with the temperature but is probably always small. We then have

$$U_{23}=\frac{a\left\{(1-\epsilon_0)u_2+\frac{b}{a'}\,u_3\right\}}{u_0\{b+1-\epsilon_0+(b+a'-a'\epsilon_0)y\}}\,. \tag{113}$$

For a hydrogen atmosphere a' is very small. When y is finite, we have

$$u_0=u_2=1-\epsilon_0 f_2(y)\,,\qquad u_3=a'(1-\epsilon_0+\tfrac{2}{3}y)\,,\qquad u=y+2-\epsilon_0\,.$$

Taking $\epsilon_0=0$, we find

$$U_{23}=U''_{23}=\frac{1+b+\frac{2}{3}by}{1+b+by}\,, \tag{114}$$

$$\frac{1}{x}=1+b+\tfrac{2}{3}by\,.$$

For the maximum,

$$by=\frac{(1+b)(I-E+vT-x\beta)}{E-\frac{1}{3}I+\frac{1}{6}vT+\frac{2}{3}x\beta}\,. \tag{115}$$

When $E>\frac{1}{3}I-\frac{1}{6}vT-\frac{2}{3}x\beta$, the maximum occurs for a finite value of by, independent of a' (to the first approximation). But if we suppose that $a'y$ is finite, we have

$$u_0=\frac{1}{1+a'y}\,,\qquad u_2=\tfrac{1}{2}+f_2(a'y)\,,\qquad u_3=u_2-u_0\,,\qquad u=y(1+a'y)\,,$$

and find, when $\epsilon_0=0$,

$$U_{23}=U''_{23}=\frac{(b+a')(1+a'y)\{\frac{1}{2}+f_2(a'y)\}-b}{a'\{1+b+(b+a')y\}} \tag{116}$$

and

$$\frac{1}{x}=1+\frac{b}{a'}\left[1-\frac{1}{(1+a'y)\{\frac{1}{2}+f_2(a'y)\}}\right].$$

When a'/b is small, (116) reduces to (105) and x is negligible. The maxima for small values of E come when $a'y$ is finite and are independent of b. Hence, if $a' << b$, the maxima of different lines of the same element may be widely separated. When $E=\frac{1}{3}I$, the maximum will be very flat.

18. In applying these formulae to actual stars, it is customary to assume a constant value of g; but it is very little trouble to allow for the variation of g along the main sequence and the giant series.

For the former, the best values of g are obtainable from eclipsing variables. All spectral types from B to M are available.[21] For the

TABLE VII

Star	Sp.	z	$\log (g/g_0)$	O−C	Star	Sp.	z	$\log (g/g_0)$	O−C
V Pup	B1	0.53	−0.54	−0.04	β Aur	A0	1.16	−0.52	−0.20
u Her	B3	.64	− .42	+ .05	TX Her	A2	1.3	− .04	+ .24
σ Aql	B3	.64	− .30	+ .17	Z Her	F5	1.7	− .29	− .11
Z Vul	B3	.64	− .51	− .04	Sun	dG0	2.0	.00	+ .09
U Oph	B8	.90	− .28	+ .12	W UMa	dG0	2.1	+ .10	− .16
RS Vul	B9	.90	− .60	− .20	α Gem C	dM0	3.3	+0.19	−0.08
TV Cas	B9	0.97	−0.52	−0.14					

brighter components of the best-determined stars the data are in Table VII. The formula

$$\log_{10}\frac{g}{g_0} = -0.65+0.28z \qquad (117a)$$

represents the data with an average residual of only ± 0.12 for the individual stars.

For the giant stars we may use Strömberg's[22] recent determinations of the mean absolute magnitude. From his Tables II and III we find the data in Table VIII, for which the average residual given by

$$\log_{10}\frac{g}{g_0} = +0.76-0.97z \qquad (117b)$$

is ± 0.07. For the natural logarithms,

$$\log g = -1.5+0.6z \text{ (dwarfs)} = 1.7-2.2z \text{ (giants)}. \qquad (118)$$

[21] Russell, Dugan, and Stewart, *Astronomy*, **2**, 719, 1927.

[22] *Mt. Wilson Contr.*, No. 442; *Astrophysical Journal*, **75**, 115, 1932.

These formulae, although purely empirical, should give a trustworthy representation of the variation in gravity along the two principal sequences of stars. Strömberg's data for the giants represent the stars brighter than the sixth magnitude, which is just the selection we need.

If now we set $d \log g = Bdz$, equation (88) becomes

$$\frac{d \log M_0}{dz} = U_2(2I + \tfrac{7}{2} vT - \alpha V + B) - (I + E + \tfrac{5}{2} vT + B) .$$

If $E + \frac{7}{4}vT + \frac{1}{2}\alpha V < \frac{1}{2}B$, $U_2 < \frac{1}{2}$ and there is no maximum—M_0 increasing to the lowest temperatures at which X' remains completely

TABLE VIII

Sp.	T	M_b	log (D/D_0)	log (m/m_0)	log (g/g_0)	z	O−C
A0	11,200	+0.6:	0.27	0.48	−0.05:	1.03	+0.18
A9	9000	+0.8	0.48	.45	−0.51	1.29	− .02
gF3	6000	+1.5	0.56	.36	−0.76	1.78	+ .20
gG3	5000	+0.3	1.03	.53	−1.53	2.32	+ .04
gK1	4000	−0.2	1.32	.60	−2.04	2.90	+ .01
gK5	3300	−1.3	1.71	.77	−2.65	3.52	+ .01
gM1	3000	−2.0	1.93	0.92	−2.94	3.86	+0.05

ionized. This state of affairs does not happen when the absorption arises from metallic ions, but it does happen when the absorption comes from hydrogen ions or neutral atoms, and αV is replaced by β or by 0.

19. We may now apply our formulae to the maxima of lines in a hydrogen atmosphere containing small amounts of ionized metals. Here $\epsilon_0 = 0$ and a' is small, so that we have $U_2 = U_2'' = \frac{1}{2} + f_2(a'y)$ and

$$f_2(a'y) = \frac{E + \frac{3}{4}vT + \frac{1}{2}\beta + \frac{1}{2}B}{2I + \frac{7}{2}vT - \beta + B} \qquad (119)$$

When the metals produce most of the ions, we must substitute αV for β.

Assuming as before $V = 2.8$ (λ 4400), we obtain Table IX, which gives data for giants and main-sequence stars and for the assumed temperatures at which the maximum occurs. The upper part of the

table gives the values of $a'y$ derived from (119), which indicate the degree of ionization of the hydrogen, and of $a'y_1''$ and $a'y_2''$, which define the extent of the first and second ionizations of the metals. For the maxima of Ti^+ and Fe^+, $a'y > 2$; most of the ions come from the metals, and αV has been used in (119). For H and Mg^+ the reverse is the case.

TABLE IX

MAXIMA AND MINIMA FOR LINES IN HYDROGEN ATMOSPHERE; OPACITY DUE TO IONS ONLY

Element	H		Ti^+		Mg^+		Fe^+	
I_1	13.54		6.80		7.61		7.83	
I_2			13.6		14.97		16.5	
E	10.15		1.16		8.83		2.80	
T	10,000	7000	10,000	7000	10,000	7000	10,000	7000
Maxima, giants:								
$a'y$	0.63	0.55	17.9	12.5	0.98	0.89	6.3	5.0
$a'y_1''$			2.2×10^{-3}	4.9×10^{-5}	3.8×10^{-3}	1.8×10^{-5}	2.0×10^{-3}	9.6×10^{-5}
$a'y_2''$			23	17	5.1	9.5	193	770
Main sequence:								
$a'y$	0.55	0.47	8.0	6.9	0.84	0.76	4.1	3.4
$a'y_1''$			9.6×10^{-4}	2.7×10^{-5}	3.3×10^{-4}	1.5×10^{-5}	1.3×10^{-3}	6.5×10^{-5}
$a'y_2''$			10	9	4.4	7.6	126	520
Minima, giants:								
y			0.13	.....	3.2	.....	0.44	.....
y_2''			0.17	.....	16.8	.....	13.7	.....
Main sequence:								
y			0.58	.....	3.8	.....	0.69	.....
y_2''			0.76	.....	19.5	.....	21.4	.....

We have, approximately, $N_0''/N_1'' = a'y''/2a'$. Hence, unless a' is very small, the proportion of neutral atoms is small, and our method is valid. Nevertheless, if $a' = 0.01$ (about the largest permissible value), 1 part in 200 of the iron is neutral, if the enhanced lines reach their maxima at 7000°, and 1 part in 10, if the maximum comes at 10,000°. The arc lines of iron are actually nearly as strong as the enhanced lines in classes F5 and F0, when the latter reach their maximum and still appear in large numbers in class A. As the observable enhanced lines correspond to transitions of small probability, this relation of intensities is what might be expected. Second ionization, even at the higher temperature, is negligible.

By (98), minima of these enhanced lines would occur, if second ionization did not intervene, when

$$y = \frac{2E'' + \frac{3}{2}vT + \beta + B}{I - E'' + vT - \beta}.$$

The corresponding values of y and y'' are given in the lower part of Table IX. Since the second ionization of *Ti* is far advanced before this condition is reached, the lines of Ti^+ will actually fade rapidly as the temperature rises. Since for *Mg* and *Fe* it is sensible, but small, there should be an inflection in the curve, if not an actual minimum.

We may now compute the difference of the temperatures at which the minima for the various lines occur. From (87), when a' is small, we have

$$\frac{2+a'y}{(1+a'y)\{\frac{1}{2}+f_2(a'y)\}} \cdot d \log y-(2I+\tfrac{3}{2}vT-\alpha V)dz-d \log g=0 .$$

The coefficient of $d \log y$ is always between 2.00 and 1.84. For hydrogen, with the assumed conditions, we have, approximately,

$$2.9\ \Delta z=1.9\ \Delta \log y-\Delta \log g=1.9\ \Delta \log y-B\ \Delta z .$$

If the maximum for Ti^+ comes at 7000° for giants, $a'y=12.5$. For the maximum of hydrogen, $a'y=0.6$, almost independent of temperature. Hence $\Delta \ln y=-3.0$, $\Delta z=-0.21$. A second approximation gives $a'y=0.58$, $\Delta z=-0.22$. Similarly, for Mg^+, $\Delta z=-0.03$; for Fe^+, $\Delta z=-0.16$. For the main sequence the differences are Ti^+, -0.17; Mg^+, -0.02; Fe^+, -0.12. For giant stars z increases by 0.55 per spectral class (from B to K). Hence the computed values of Δz indicate that the maxima of Fe^+ and Ti^+ should occur 0.3 and 0.4 classes later than those of *H* and Mg^+. The observed maxima are Ti^+, F5–Go; Fe^+, F5; Mg^+, A2; *H*, Ao—in the same order, but much farther apart.

Part of this discrepancy may arise because almost half of Miss Payne's[23] F5 and G stars are super-giants. If g is one-tenth as great for these as for ordinary giants, Δz would be changed by -0.08. Again, the spectral energy-curve of A stars is disturbed by hydrogen absorption—being relatively low in the red, beyond the limit of the Paschen series, and high in the violet—so that the color index gives too high a temperature. McCrea's calculations[24] show that, for stars with atmospheres of pure hydrogen and temperatures of 8000° and

[23] *Stellar Atmospheres*, pp. 127–130, 1925.

[24] *Op. cit.*, p. 855.

10,000°, the slope of the energy-curve in the visible spectrum corresponds approximately to temperatures of 12,000° and 10,000°, so that this may account for the whole discrepancy.

20. If, however, the neutral hydrogen atoms produce much more opacity than the metallic ions, we must use (111) for H and Mg^+, and (105) for Fe^+ and Ti^+, with the results given in Table X. We have

TABLE X

MAXIMA IN HYDROGEN ATMOSPHERE; OPACITY DUE TO NEUTRAL ATOMS AND IONS

Element	H		Mg^+			Ti^+		Fe^+	
T	10,000	7000	10,000	7000	T	10,000	7000	10,000	7000
Giants:					Giants:				
U_3	0.855	0.865	0.806	0.814	U_3	0.525	0.519	0.585	0.580
by	0.77	0.73	1.40	1.26	$a'y$	14.4	20.2	2.20	2.60
Z	2.35	2.33	2.48	2.46	Z	2.03	2.02	2.26	2.20
Main sequence:					Main sequence:				
U_3	0.868	0.879	0.823	0.833	U_3	0.569	0.564	0.622	0.619
$by/(1+b)$	0.66	0.57	1.14	1.00	$a'y$	3.40	3.88	0.76	0.82

now, from (107), setting $\epsilon=0$,

$$d\log(\tau_1+\tau_2) = -(2I+\tfrac{7}{2}vT-x\beta)dz - d\log g + Z\,d\log y\,,$$

where

$$Z=\frac{a'Y\{1+b+(b+a')y\}}{a'u_2+bu_3}\,.$$

If a' is small, and y and b finite,

$$Z=\frac{2(1+b+by)}{1+b+\frac{2}{3}by}\,;$$

but if $a'y$ is finite, and a'/b' small,

$$Z=\frac{2+a'y}{(1+a'y)U_3''}\,,$$

where U_3'' is given by (104). Hence $Z=2$ when by is small, rises to 3 when by is large and $a'y$ small, and drops to 2 when $a'y$ is small. Its values at the maxima are given in Table X.

Comparing the maxima of H and Mg^+, we have (closely enough) for giants $-27\,\Delta z+2.4\,\Delta\log y=0$, whence $\Delta z=-0.05$. That is, the maximum for H comes 0.1 spectral class earlier than for Mg^+. Similarly for Fe^+ and Ti^+ (near 7000°), $\Delta z=-0.16$, the maximum for Fe^+ coming 0.3 class earlier. Comparing Fe^+ (at 7000°) and H, we have $\Delta\ln y=-1.1+\ln(a'/b)$. In this case Z rises to 3 in the interval, so that we may adopt a mean value of 2.7 and obtain $\Delta z=-0.11+0.1\ln(a'/b)$. The observed separation is about $1\frac{1}{2}$ spectral classes; $\Delta z=-0.8$, which gives $a'=0.001b$. With $a'=0.01b$, we would have $\Delta z=-0.57$. In view of the corrections already mentioned and the uncertainty of the observed maxima, no definite determination of a'/b can be made.

21. These results may be compared with those predicted for an atmosphere composed of some element entirely in the first state of ionization (e.g., Na) with traces of H, Fe^+, etc. In this case we have, from (46),

$$\mu g N_0=2\epsilon K\{y''-\log(1+y'')\},$$

and from (62), with $\epsilon=1$,

$$\mu g\tau=C_1\lambda^3T^{-\frac{3}{2}}FIK^2y^2,$$

in which we have set $G=0$, since there are supposed to be no neutral atoms of the active element.

For the maximum we find easily

$$\frac{E''+\frac{3}{4}vT}{I''+E''+\frac{5}{2}vT}=2f_1(y'')-f_2(y'').$$

This is equivalent to the equation (38) used by Milne and Chandrasekhar[25] in discussing maxima.

Only the atomic constants for the element under consideration appear, for obvious reasons. The lines have maxima, but no minima. In Table XI, the entries headed y show the ionization of hydrogen when the other lines are at their maxima. The values of Δz (com-

[25] *Op. cit.*, pp. 180–183.

puted as before) show that, in a once-ionized atmosphere, the maxima for Fe^+ should come at the same temperature as for Mg^+, and higher than for H. This is contrary to fact. A still more serious discordance is indicated by the values of y_1''. At the observed temperature of maximum for Fe^+, only one iron atom in a million should be neutral. As a matter of fact, the arc lines of the metals remain as high as class A. In the spectrum of Sirius, 248 arc lines of various metals have been observed, and only 76 enhanced lines.[26] This is conclusive evidence of the great abundance of hydrogen.

The computed maxima for Fe^+ and Ti^+ occur when second ionization is moderate, and the maximum for Mg^+ when it is large, on account of the high value of E.

TABLE XI

MAXIMA OF LINES IN SODIUM ATMOSPHERE

	H		Ti^+		Mg^+		Fe^+	
T....	10,000	7000	10,000	7000	10,000	7000	10,000	7000
y_2''...	.....	.....	20.2	22.8	1.29	1.28	10.1	10.7
y_1''...	0.65	0.62	2.0×10^{-3}	1.0×10^{-4}	6.3×10^{-5}	1.9×10^{-6}	6.4×10^{-4}	2.3×10^{-6}
y....	.65	.62	16.0	17.4	0.25	0.12	0.32	0.08
Δz...	0.00	0.00	+0.22	+0.23	−0.07	−0.11	−0.05	−0.14

22. The maxima of spectral lines are complicated phenomena, often depending more on the properties of other elements than on those of the element that produces them. Moreover, they can rarely be determined with accuracy from existing observational data. Some better way of determining electron pressures in stellar atmospheres is to be desired. The most trustworthy data appear to be those for the general level of ionization in the solar atmosphere derived by the writer,[27] and in sun-spots, by Miss Moore.[28] These depend upon a general calibration of hundreds of lines. The conclusion of Minnaert and Wanders[29] that the atmosphere above the spots is in radiative equilibrium fully justifies their use as independent data.

[26] Adams and Russell, *Mt. Wilson Contr.*, No. 359; *Astrophysical Journal*, **68**, 9, 1928.

[27] *Mt. Wilson Contr.*, No. 383; *Astrophysical Journal*, **70**, 54, 1929.

[28] *Mt. Wilson Contr.*, No. 446; *Astrophysical Journal*, **75**, 321, 1932.

[29] *Zeitschrift für Astrophysik*, **5**, 297, 1932.

For the disk, $T=5740°$ and $I_0=8.5\pm0.1$. For this value of I, $N_0=N_1$, and $y=2$; whence $K=23$, $P=46\pm9$. For the spots, $T=4720°$, $I=7.0\pm0.1$, $P=28\pm7$.

We shall now assume, as the best convenient approximation, that the solar atmosphere contains in addition to completely ionized and neutral gases a single "active" or partly ionized element. If the opacity is produced by ions, we have, by (83),

$$\mu' M_0 g \tau_1 = C_1 \lambda^3 T^{-\frac{3}{2}} G(\lambda, T) P^2 u_2 . \qquad (120)$$

Here μ' is the mean atomic weight in chemical units and $M_0=1.649\times10^{-24}$, the mass of this unit. For the sun, $g_0=2.74\times10^4$. Setting as before $\lambda=4400$, we find for the disk $G(\lambda, T)=741$, and

$$\mu'\tau = 2.33\times10^{-5}, \qquad P^2u_2 = 0.050u_2 .$$

For the spot, $G=2400$, $\mu'\tau=0.081u_2$.

So far, we have made no specific assumption regarding the composition of the atmosphere, but we must do so to find u_2. Since hydrogen is in great excess, we may set $\epsilon_0=1$ in (83), obtaining $u_2=\frac{1}{2}+f_2(a'y)-f_2(y)$, so that u_2 is always between $\frac{1}{2}$ and 1. About three-quarters of the metallic atoms in the sun[30] are of *Mg*, *Fe*, and *Si*, with a mean I of 7.8 volts. Most of the rest are of *Ca*, *Al*, *Na*, and *K*, and are almost completely ionized, even in spots.[31] We therefore adopt $a'=0.25$, $I=7.8$, and find for the disk, $K=95$, $y=0.49$, $u_2=0.58$, $\mu'\tau=0.029$; for the spots, $K=1.96$, $y=14.5$, $u_2=0.61$, $\mu'\tau=0.050$. Since hydrogen is in great excess, μ' must be nearly 1. Even so, the computed values of τ_1 are about one-tenth as great as might have been expected. No permissible changes in the assumed solar data can do much to relieve this discordance. This may arise from imperfections of the approximate theory;[32] but it is just what was to be anticipated if neutral hydrogen atoms produce most of the opacity, and it may be regarded as evidence in favor of this hypothesis.

[30] H. N. Russell, *Mt. Wilson Contr.*, No. 383; *Astrophysical Journal*, **70**, 56, 1929.

[31] C. E. Moore, *Mt. Wilson Contr.*, No. 446; *Astrophysical Journal*, **75**, 322, 1932 (Table VIII).

[32] Professor D. H. Menzel has very kindly informed me that his unpublished work indicates that the value of τ given by Pannekoek's formula should be considerably increased.

Certain additional tests are available. Miss Moore, comparing the disk and spot, finds, in our present notation,

$$\frac{P'}{P} = 0.60 \pm 0.10 , \qquad \frac{N'}{N} = 1.70 \pm 0.11 ,$$

the primes referring to the spot. Now, by (80),

$$\mu' M_0 g N = aPu . \tag{121}$$

With the data already adopted, $u' = 3.36$, $u = 1.33$, whence $P'/P = 0.40N'/N = 0.68$. The agreement is within the probable error.[33] For the number of atoms per square centimeter above the photosphere we find $\mu' N = 1.02 \times 10^{21}$. This does not include the non-ionized material. From direct calibration of solar lines, with the aid of Unsöld's results, the writer[34] finds 8×10^{20}.

These results depend only on the assumption of an isothermal atmosphere, and not on the law of opacity. The latter determines the change of P and N with temperature. If $\mu =$ constant, $\tau_1 =$ constant,

$$P^2 \propto \frac{gT^{\frac{3}{2}}}{u_2 G(\lambda, T)} . \tag{122}$$

If $\tau_2 =$ constant,

$$P^2 \propto \frac{a'(1-\epsilon_0) g T^{\frac{3}{2}}}{u_3} , \tag{123}$$

where $a'(1-\epsilon) = \epsilon'$, the fraction of the atoms which are completely ionized to the lowest temperatures considered.

With the values just derived for the sun as standards, those for any other star may be computed. Since u is a function of y, and therefore of P, the solution must be made by approximations, which converge rapidly.

On introducing the solar values, (122) gives

$$\log P = +0.16 + \tfrac{1}{2} \log \frac{g}{g_0} + \tfrac{3}{4} \log \tau - \tfrac{1}{2} \log u_2 - \tfrac{1}{2} \log G(\lambda, T) . \tag{124}$$

[33] Cf. the more detailed calculation (*loc. cit.*).

[34] *Mt. Wilson Contr.*, No. 383; *Astrophysical Journal*, **70**, 63, 1929.

Then, for the spot,

$$\log P' = 1.23 + \tfrac{1}{2} \log u_2' .$$

If we assume for the first approximation that $u_2' = 0.58$, as for the disk, $\log P' = 1.35$, $P' = 22.2$; but $K' = 1.96$, $y' = 11.4$. Hence, $u_2' = 0.616$, which gives $P' = 21.6$, $y' = 11.1$. This yields $u_2' = 0.618$, $P' = 21.6$. The third approximation is superfluous.

We now have $u' = 3.20$, and by (121), $N'/N = 1.12$. With atomic opacity,

$$P \sqrt{u_3} \propto T^{\frac{3}{4}} (\epsilon' g)^{\frac{1}{2}} . \qquad (125)$$

For the disk, $u_3 = 0.179$. For the spot, y is large; and, as a first approximation, we may set $u_3' = \tfrac{1}{2}$; whence, $P' = 20.5$, $u_3' = 0.431$, $P' = 22.0$. The third approximation gives $P' = 21.8$, $y' = 11.1$, $u_3' = 0.442$, and $N'/N = 1.14$.

The agreement of the results of the two assumptions is a coincidence. The computed values of P agree tolerably with those observed; those of N'/N are seriously discordant. With a large value of a', that is, with more of the easily ionized metals, the agreement on the second hypothesis would be better.

23. We may now determine a' from the observed maxima of Fe^+ and Ti^+ (Table XII). For various assumed temperatures of maximum, we calculate $P \sqrt{u_2}$ from (124), taking $\log (g/g_0)$ from (118) for giant stars, and also K for hydrogen. Then from (119) we find the values of $a'y$ which make the line under consideration a maximum at this temperature. The values of u_2, y, and a follow at once. If the opacity is due to neutral atoms, we may use (125), remembering that, on our assumptions, $\epsilon' = a'$ when the hydrogen is being ionized, but $\epsilon' = \tfrac{1}{4} a'$ in the sun. We then obtain the results given in the lower part of Table XII.

The observed maximum for Fe^+ is in class F5, for which, by Table VIII, $z = 1.88$; whence, if the opacity is due to ions, $a' = 0.001$. For Ti^+ it is at F8 ($z = 2.05$) and $a' = 0.0005$. With opacity due to neutral atoms, $a' = 0.0004$ in both cases.

A change of 0.2 spectral class in the assumed position of maximum (which is by no means improbable) changes a' by a factor of 4.

TABLE XII

DETERMINATION OF a', GIANT STARS

T	6000°	6500°	7000°	$Fe+$ T	6000°	6500°	7000°	$Ti+$ 6000°	6500°	7000°
	Opacity Due to Ions									
z	1.94	1.79	1.66	$a'y$	4.50	4.70	4.95	10.8	11.6	12.5
$\log (g/g_0)$	−1.12	−0.97	−0.85	u_2	0.639	0.635	0.630	0.571	0.567	0.564
$PV u_2$	11.2	16.8	23.6	y	8900	1470	300	9400	1550	320
K	0.0016	0.0144	0.097	$\log a'$	−3.30	−2.50	−1.78	−2.94	−2.13	−1.41
	Opacity Due to Neutral Atoms									
$PV\sqrt{a'u_3/\epsilon}$	15.6	19.6	23.7	$a'y$	2.63	2.57	2.50	22.1	27.9	33.7
				u_3	0.419	0.417	0.413	0.495	0.494	0.494
				y	15300	2100	370	14100	1930	340
				$\log a'$	−3.77	−2.91	−2.17	−2.80	−1.96	−1.24

There is also some uncertainty regarding the value of gravity, but this is less serious. According to Strömberg, super-giant stars of classes F and G average about four magnitudes brighter than normal giants. If they obey the mass-luminosity law they should have about six times the mass of normal giants and eight times the diameter (allowing something for lower temperature), so that g should be about one-tenth the assumed value. If all the observed stars were super-giants, the tabular values of a' would have to be multiplied by about 3. Any permissible correction must be much smaller.

The conclusion that a' is of the order of 0.001, that is, that hydrogen is about a thousand times as abundant as all the metals together, appears to be well founded.

The strength of the Balmer lines in the sun indicates a still greater abundance of hydrogen, but, as is well known, the determination is vitiated by the Stark effect. Unsöld, by comparison of the intensities of the (K) line and the continuous absorption beyond the Balmer limit (which should be free from disturbance by the Stark effect), finds for class A2, $\log_{10} N_H/N_{Ca} = 4.2$, but reduces this to 3.0 by an empirical allowance for departure from thermodynamic equilibrium. Since calcium contributes about 4 per cent of all the metallic atoms,[35] this makes $a' = 0.0016$ without the empirical correction, and 0.025 with it. The reality of the "correction," which was suggested four years ago by the writer, now appears very doubtful.

Menzel's careful study of the chromosphere[36] leads to $a' = 2 \times 10^{-6}$. This value involves an empirical correction for departure from thermodynamic equilibrium.

24. The relative absorbing power of ionized and neutral hydrogen atoms may now be estimated from the maxima of H or Mg^+ by (106) and (111) with $\epsilon_0 = 0$:

$$a'\mu g(\tau_1+\tau_2) = C_1\lambda^3 T^{-\frac{3}{2}} P^2 u_{23}; \qquad u_{23} = Hu_2 + Au_3 .$$

Now $bH = a'A$. If $b \gg a'$ and $a'y$ is finite, we may set $u_{23} = u_3$. But if y is finite,

$$u_2 \backsimeq 1, \qquad u_3 \backsimeq a'(1+\tfrac{2}{3}y) ,$$

[35] H. N. Russell, *Mt. Wilson Contr.*, No. 383; *Astrophysical Journal*, **70**, 56, 1929.

[36] *Lick Observatory Publications*, **17**, 281, 1931.

so that

$$\frac{u_{23}}{u_3} = \frac{1+b+\frac{2}{3}by}{b+\frac{2}{3}by}.$$

We then have

$$P^2u_3 = a'K^2(y^2+\tfrac{2}{3}y^3) \propto \epsilon' T^{\frac{3}{2}} g \frac{b'+\frac{2}{3}b'y}{1+b'y},$$

where $b' = b/(1+b)$. Since $b'y$ is determined by the condition for the maximum, the last expression is approximately known. If a' has been determined as above, y may be found for any given temperature, and then b', then a better value of y, and so on, with rapid convergence.

Table XIII gives the results for giant stars, on the assumption that $a' = 0.0004$. When y is large, $b' \propto a^{\frac{1}{3}}$. Computation shows that

TABLE XIII

T_{max}	K	P^2u_3	H			Mg^+		
			$b'y$	y	b	$b'y$	y	b
10000	200	3010	0.79	4.63	0.21	1.41	5.15	0.38
9000	26.7	1920	.77	15.1	.054	1.37	16.9	.088
8000	2.21	1090	0.73	65.7	0.011	1.33	74.4	0.018

this relation is very nearly true up to $a' = 0.04$. The uncertainty of the precise spectral type at which the maximum occurs, and that of the actual temperature of A stars, is too great to permit even an attempt to evaluate b; but it is clear that there is abundant latitude to satisfy our assumed condition, $a' \ll b \ll 1$.

25. The degree to which existing theories agree with observation cannot be well tested by computations from assumed maxima. It is much better to start with some definite assumption simple enough to be workable, compute the electron pressure "at the photosphere" and the quantity of gas above it by the formulae already given, and then calculate the number of atoms which should be at work in a given line. Since the importance of opacity due to neutral hydrogen atoms is still uncertain, it is assumed that the opacity arises entirely

from encounters of ions and electrons, and that, at a given temperature, only one element is "active" or partly ionized, the rest being completely ionized or neutral. Our fundamental equations then are

$$(1-\epsilon_0)\mu g N = P\left\{\frac{ay}{1+a'y}+2-\epsilon_0\right\} = Pu\,, \tag{126}$$

$$\log P\sqrt{u_2} = 0.16+\tfrac{1}{2}\log\frac{g}{g_0}+\tfrac{3}{4}\log\tau-\tfrac{1}{2}\log G\,(\lambda, T)\,, \tag{124}$$

$$u_2 = \tfrac{1}{2}(1+a'-a'\epsilon_0)+f_2(a'y)-(a'+a\epsilon_0)f_2(y)\,, \tag{127}$$

$$\log\frac{g}{g_0} = 0.76-0.97z \text{ (giants)}\,, \tag{117b}$$

$$= -0.65+0.28z \text{ (main sequence)}\,, \tag{117a}$$

$z = 11{,}600/T$; $y = P/K$. Logarithms are here to the base 10. Equation (80) has been modified so that N represents the number of *all* atoms, not that of atoms of the active element, as before. ϵ_0 is the fraction of non-ionized atoms in the whole atmosphere; a and a' those of partly and completely ionized elements among the rest. Equations (117*a*) and (117*b*) are empirical (§18); the numerical constant in (124) depends on solar data (§22). When most of the ions come from the hydrogen, $G\,(\lambda,T)$ in equation (11) should be replaced by $H(T)$, equation (12). The composition of the atmosphere is assumed to be 999 parts hydrogen to 1 of metals ($a' = 0.001$). Of the metals, three-fourths are iron, three-fourths of the rest sodium, and the remainder potassium. This gives a tolerable representation of the composition of the sun, without being too complicated to work with. Helium is neglected, since, according to Unsöld's determination (the best yet available),[37] its abundance is only 1 per cent that of hydrogen. Oxygen has so nearly the same I.P. as hydrogen that it may be combined with it. Second ionization is neglected, since it affects the whole number of free electrons by 1 per cent at most. The field of temperature falls into several ranges, in each of which one of the elements undergoes ionization. Within each range, a' and ϵ_0 are constant, so that u and u_2 are functions only of y, and they or their

[37] *Zeitschrift für Astrophysik*, **3**, 99, 1931.

logarithms may be plotted, and the values of K (for the active element) and G (or H) tabulated for various values of T or z, as also that of $\log P\sqrt{u_2}$. We then have $\log y\sqrt{u_2}$ at once, and $\log y$ almost by inspection, and then $\log P$ and $\log N$.

The results for the main sequence are shown in Figure 1, in which $\log N$ is plotted against $z' = z \text{ Mod} = 5040/T$. The curves marked H, Fe, Na, and K show how $\log N$ or $\log P$ would vary in an atmosphere containing the assumed proportions of the given element in which

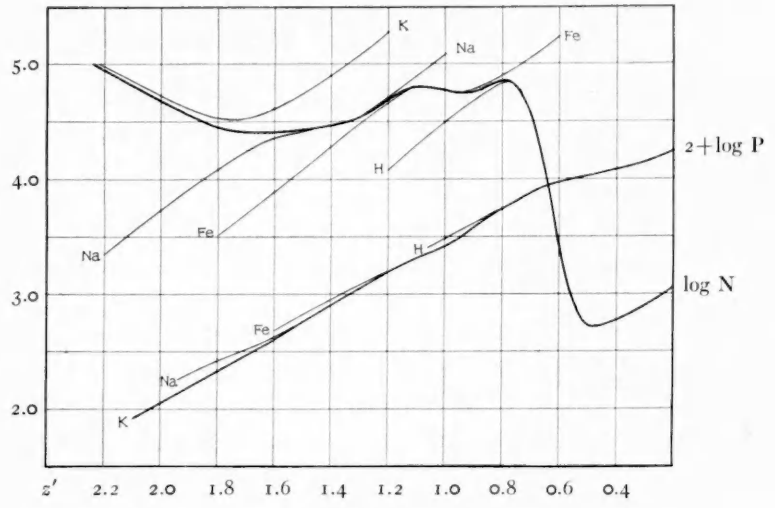

FIG. 1

the other elements were completely neutral or wholly ionized under all conditions. The nearly straight parts of the curves, sloping upward to the right, correspond to temperatures at which the active element is practically all neutral or all ionized; the intermediate part, with the inflection, corresponds to its ionization. The final part of one curve and the initial part of the next are asymptotic to curves representing an atmosphere containing only completely ionized and completely neutral atoms in the appropriate proportions. These curves are slightly concave upward. The curves for H and Fe, and again for Fe and Na, approach very closely, showing that the ionization of one is almost complete before that of the next begins. This is not the case for Na and K. The course of the curve when

both are undergoing ionization has been calculated with the aid of equations (54), (58). The other curves approach so closely that the transition can be drawn in freehand. The rise at the lowest temperatures ($z' > 1.8$) results from the assumption that no other source of opacity is present besides the interaction of the few electrons and ions which come from the potassium. When $z' = 2$, only 1 in 10^5 of all the atoms is ionized, and opacity due to other sources probably exceeds that here considered. This would flatten out the rise at lower temperatures and might reverse it.

A complication arises in the curve for hydrogen. With very small ionization almost all the free ions are metallic, and the function G must be used in the equation. With high ionization, the hydrogen predominates and H is to be used. The number of metallic ions is $a'N$, while that of the hydrogen ions is N_1. The contributions of the two to the opacity are proportional to $Ga'N$ and HN_1, or, by (80) and (81), to $aa'Gu$ and Hu_1. In the region of transition, where the two are comparable, G and H are not very different, and a linear interpolation between the values computed with each separately gives accurate enough results.

The constant 2 has been added to log P in Figure 1 to get it into the picture. Here the curves for *Fe* and *Na* show shallow depressions due to the change of u_2 from 0.50 for both high and low ionization to a maximum of 0.66, when $y = 3.5$. Those for K and H drop lower, since u_2 rises to 1. The net effect of all these influences is that log P varies almost linearly with z' except at the highest temperatures.

Figure 2 gives similar data for giant stars having the assumed relation between surface temperature and gravity. The change of log P with temperature is here very rapid, and still more nearly linear. The empirical expression $\log P = 3.46 - 2.74z'$ represents the values for $z' = 2.40$ to $z' = 0.50$ with an average residual of only ± 0.025. At this temperature the giants run into the main sequence, and to use the formula beyond this limit is illusory. The values of N for the giants show an even greater drop, while the hydrogen is being ionized, and increase gradually toward lower temperatures, while those for the main sequence decrease. It is now evident why the assumption of an atmosphere of constant extent gives in many in-

stances a tolerable approximation. The mutual interaction of opposing factors causes the amount of material "above the photosphere" to vary relatively little from $z'=0.8$ ($T=6300°$) to the lowest temperatures. The amount is again nearly constant from 10,000° upward. The rudimentary theory based on a constant opacity coefficient and extent of atmosphere therefore gives a fairly good representation of the effects of temperature on line intensity from classes M to F5 and again from B9 upward. It fails in the intermediate re-

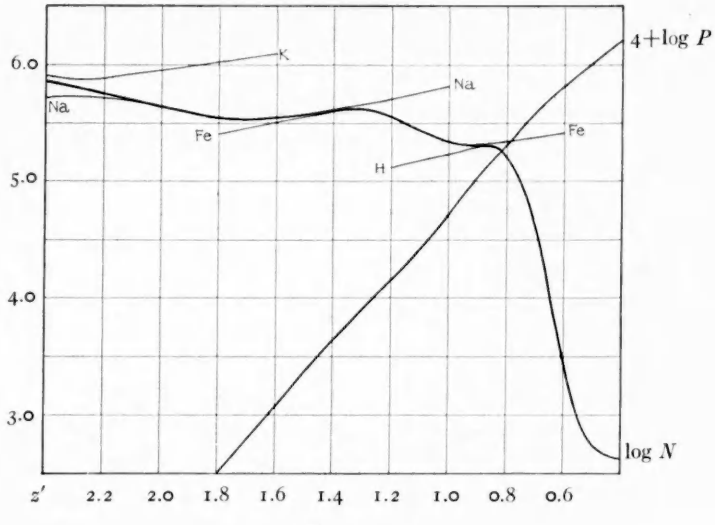

FIG. 2

gion and does not account for the absolute-magnitude effects even in the cooler stars.

26. We are now in a position to calculate the number of atoms engaged in the production of any given line. The number of atoms of the given element per unit area above the photosphere is $N''=N\epsilon''$. For those of neutral or ionized atoms, we may use Milne's approximation and set $N_0''=N''y/(2+y)$, $N_1''=2N''/(2+y)$. The error of this approximation is unimportant for our present purpose, except when ionization is just beginning in the presence of other fully ionized material, that is, for ultimate enhanced lines near their first appearance, and no such lines are observable, unless perhaps in the fainter red dwarfs. When second ionization is sensible,

we have $N = N_0(1+\frac{1}{2}y_1+\frac{1}{4}y_1y_2)$. Since $y_1 \gg y_2$, we may set $N = N_0(1+\frac{1}{2}y_1)(1+\frac{1}{2}y_2)$ with no sensible error.

The number of atoms in the appropriate excited state is $M_0'' = N_0'' qe^{-Ez}/B$; but only a certain fraction, f, of these will be engaged in the production of a given line. For the leading members of strong series, f is nearly unity; for improbable, though permitted, transitions, it may be very small. Values have been calculated by wave mechanics in a few of the simplest cases, such as the Balmer series.[38] For other strong lines it may often be estimated, e.g., by Kronig's rules for intensities in triads. For weak lines it may be found empirically. If L is the number of atoms actually at work on the line, we have for arc lines

$$\log_{10} L_0'' = \log N + \log \epsilon'' + \log \frac{qf}{B} + \log \frac{y_1}{2+y_1} + \log \frac{y_2}{2+y_2} - Ez' .$$

The abundance factors ϵ'', which have been taken from the writer's work on the sun's atmosphere, are proportional to the quantity called T in Table XIV.[39] From this table, $\log T = 8.1$ for all the metals (including silicon). With $a' = 0.001$, $\log T = 11.1$ for hydrogen. Then $\log \epsilon = \log T - 11.1$. The partition functions B have been taken as constant, which again introduces no significant error. The weights q apply to the individual lower energy state for the line, except for unresolved blends, when the whole weight of the blend is taken. Reasons for the assumed values of f are given in the notes to Table XIV, which summarizes the data for a number of important lines.

Figure 3 shows the predicted behavior of $\log L$ for the metals and hydrogen in the main sequence, and Figure 4 in giants. The shapes of the curves reveal the processes which are at work. For example, the curve for Fe^+ in Figure 4 rises steeply at low temperatures (near $z' = 1.3$) because the iron is rapidly becoming ionized. The smaller slope beyond $z' = 1.1$ corresponds to the Boltzmann factor, practically all the iron being ionized. The maximum at 0.8 and the ensuing drop result from the increase of opacity due to ionization of the hydrogen. As this comes to an end, the curve eases off and would turn upward again were it not for the second ionization of the metal, which causes

[38] Y. Sugiura, *Scientific Papers, Tokyo*, **11**, 1, 1929.

[39] *Mt. Wilson Contr.*, No. 383; *Astrophysical Journal*, **70**, 56, 1929.

the final rapid drop. The predicted behavior of the metallic lines agrees closely with observation. As an illustration, the maxima

TABLE XIV

DATA FOR INDIVIDUAL LINES

El.	λ	log ϵ	I	B_0	B_1	E	q	log f	log (qf/B)	Notes
H	4340	0.0	13.54	2	1	10.15	8	−1.35	−0.75	1
He	4471	−2.0	24.48	2	2	20.87	8	−1.2	−0.6:	3
He^+	4686	.....	54.16	2	1	48.16	18	−0.25	+0.7:	2
O	7772	−2.0:	13.56	9	4	9.11	5	−0.1	−0.3	2
O^+	4649	.....	34.94	4	9	22.90	6	−0.6	−0.3	5
Mg	5183	−3.3	7.69	2	2	2.67	5	−0.6	−0.2	4
Mg^+	4481	.....	14.97	2	1	8.83	10	−0.1	+0.6	2
Si	3905	−3.8	8.12	9	6	1.90	1	−0.15:	−1.1:	2
Si^+	4130	.....	16.27	6	2	9.79	6	−0.2	−0.2	2
Si^{++}	4552	.....	33.30	2	2	18.92	3	−0.1	0.0	2
Si^{+++}	4088	.....	44.95	2	1	23.95	2	−0.1	−0.1	2
Ca	4226	−4.4	6.09	2	2	0.00	2	−0.1	−0.1	2
	4554	.....	.....	.....	.....	1.88	5	−1.4	−1.0	3
Ca^+	3933	.....	11.82	2	1	0.00	2	−0.2	−0.2	2
Fe	4383	−3.9	7.83	45	60	1.48	9	−0.8	−1.5	6
Fe^+	4583	.....	16.9	60	30	2.80	10	−2.3	−3.1	7
Zn	4810	−6.2	9.36	2	2	4.06	5	−0.6	−0.2	4
Zn^+	.....	.....	17.89	2	1	.....	.....	.....	.....	.....

NOTES TO TABLE XIV

1. Calculated from wave mechanics.
2. First line of strongest series from the given limit.
3. Second line of strongest series.
4. First line of weaker of the two subordinate series.
5. Leading line of strongest multiplet of principal triad.
6. One of four strong lines of nearly equal intensity from this level.
7. Combination between terms having different limits in Fe^{++}. The strong multiplets from this level are in the far ultra-violet. The total intensity of the observable solar lines from this level corresponds to log $M = 2.6$ as against 4.6 predicted for all lines. Hence log f is about -2.0 for all the visible lines and -2.3 for the strongest from the given component level.

For the higher ionizations, the columns headed B_0 and B_1 give the values for the lower and higher states involved.

In this table the values of q for states involving two equivalent s-electrons (such as the normal states of He, Mg, Ca) have been doubled. I am indebted to Professor E. U. Condon for information that recent developments in wave mechanics show that this should not have been done. As it was done, however, in the calculations of the abundance of the elements in the sun, on which the present computations are based, the data are at least self-consistent and satisfactory for illustrative purposes, which is the object of the present diagrams. A recalculation would affect the computed maxima by much less than the errors of observation.

for the giant series are shown in Table XV. The values of z' have been translated into spectral type with the aid of Table VIII. The observed values, from Miss Payne's *Stellar Atmospheres*,[40] usually agree as well as might be anticipated from the character of the data, except that $H\gamma$ and Mg^+ come at too low a temperature (see below). For Ca^+, on the other hand, the maximum in

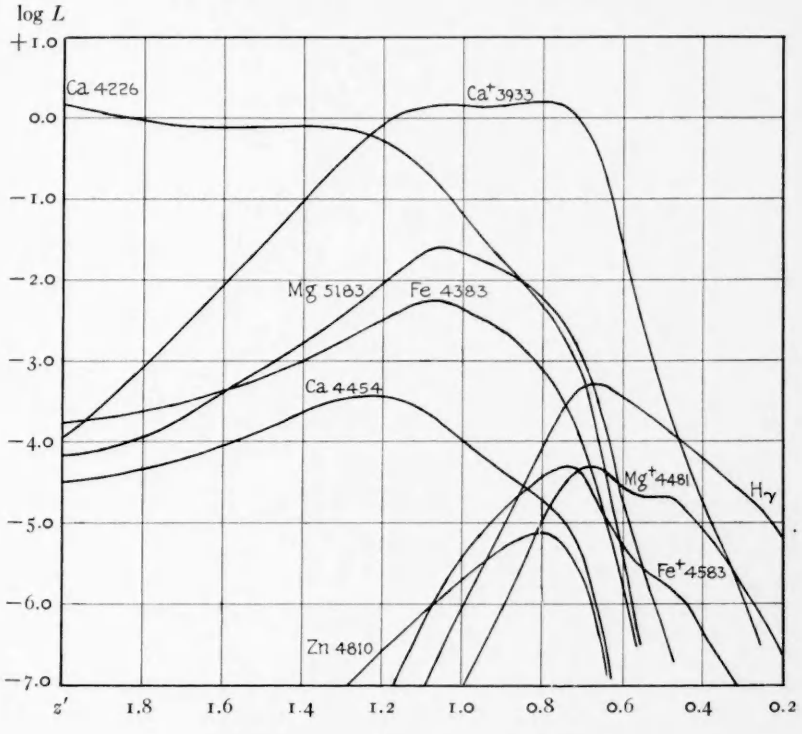

FIG. 3

class K, which has hitherto been a great puzzle,[41] is explained. What was overlooked in the earlier theories was simply the fact that the maximum for an ultimate enhanced line for an atmosphere of uniform depth is very flat-topped. For the giants, the diminished pressure and opacity at low temperatures throw the maximum to the point where first ionization is nearly ended; for the main sequence,

[40] Pp. 127–130.

[41] Cf. Milne, *op. cit.*, **92**, 183, 1932.

the opposite change in gravity throws it to the onset of second ionization, in class F5, with very little change down to K0.

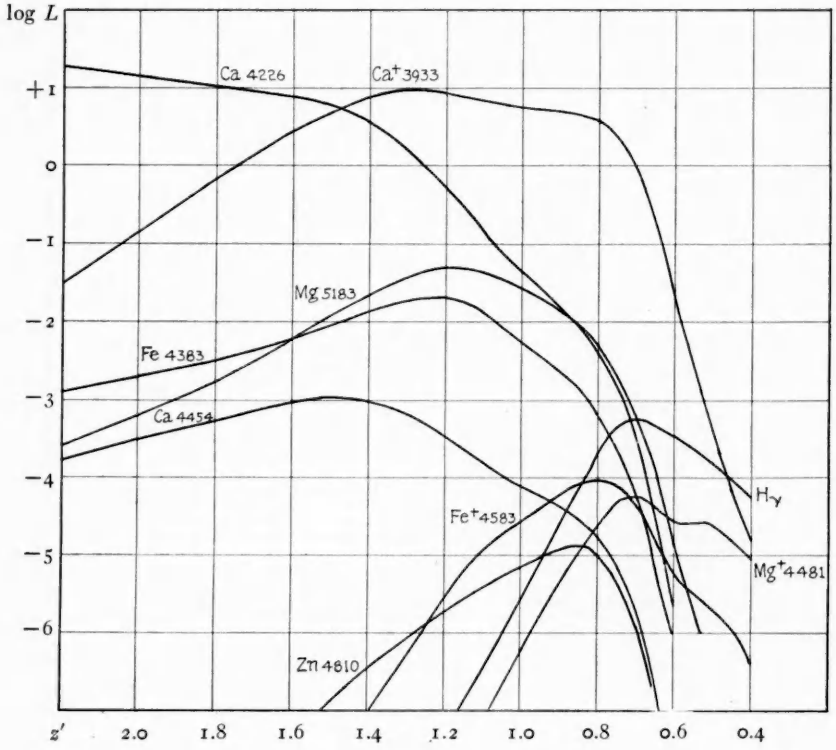

FIG. 4

TABLE XV

El.	λ	z′	T	Sp.	Obs.
Hγ	4341	0.70	7200°	F0	A0
Mg+	4481	0.71	7100	F0	A2
Fe+	4583	0.80	6300	F4	F5
Zn	4810	0.85	5900	F6	G0
Mg	5183	1.20	4200	G9	K2?
Ca+	3933	1.28	3920	K1	K0
Fe	4383	1.24	4050	K0	K2
Ca	4454	1.50	3350	K7	Ma
Ca	4226	.....	.....	None	None

The relative values of log L for $z = 0.88$ are of course in agreement with the solar data from which they were derived; but at other temperatures there is unforced agreement with observation; for example, the approximate equality of the maximum values of log L for Fe^+ and Mg^+. Miss Payne finds a maximum intensity of 6.0 for the latter and of 9.0 for other Fe^+ lines comparable in strength with the former. The latter figure must be raised by the inclusion of supergiant stars.

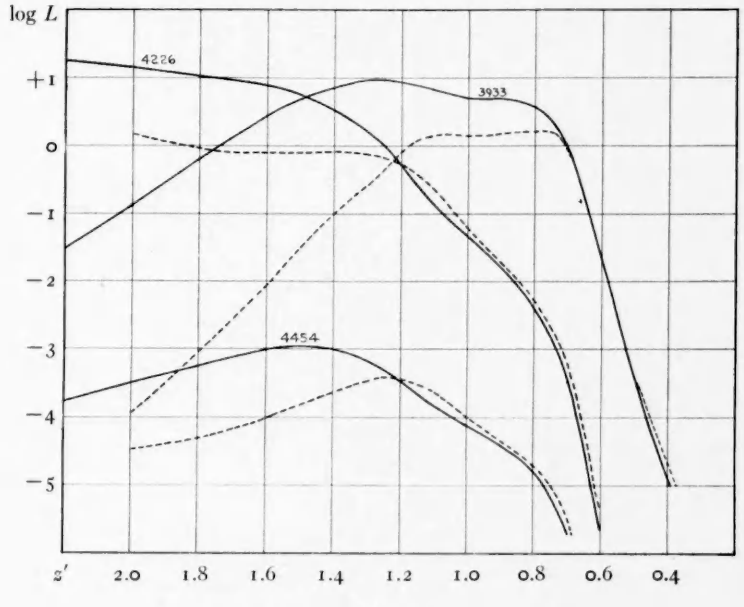

FIG. 5

For hydrogen the well-known and enormous discrepancy appears, the computed maximum L for $H\gamma$ being only ten times greater than for λ 4481, while the actual strength indicates a value fully a thousand times greater. This can hardly be attributed, even in part, to an abnormality in the Boltzmann factor, since λ 4481, for which E is seven-eighths as great, agrees very well with prediction. It is doubtless due to the "pressure effect" of the fields of neighboring ions, as discussed in detail by Unsöld. The values for calcium in giants and the main sequence are compared in Figure 5. The enhanced line (K) is always stronger in giants. At low temperatures, the difference

should be very conspicuous, and it is actually so in class M. The arc lines also are predicted to be stronger in giants than dwarfs, below $z' = 1.2$, corresponding to gG8 and dK3. For temperatures above this value the intensity should be greater in the dwarfs, if the comparison is made between stars of the same temperature; but for stars of the same spectral class the temperature difference very nearly counteracts this inequality. This prediction could well be tested by observation with high dispersion. It has, of course, no bearing on determinations of absolute magnitude from the relative intensities of neighboring lines of different sorts.

The predicted maxima for Fe^+ and Mg^+ are much too near together. It was shown in §20 that the assumption of opacity due to neutral atoms would correct this. By (106) the total opacity is thus proportional to $Gu_2 + Au_3$ (when metallic ions predominate). In §22 it was found that the term Gu_2 accounted only for an opacity $\mu'\tau = 0.029$ in the sun, while the actual value is probably about 0.4. With $G = 741$, $u_2 = 0.58$, $u_3 = 0.18$, this involves $A = 31{,}000$. Now $A = C_2/C_1\epsilon'$ and, with our assumed data, $\epsilon' = 1/4000$. Hence $C_2/C_1 = 7.9$. The opacity due to hydrogen ions alone is proportional to $80\,C_1$; that due to atoms, to C_2, so that to represent the solar data, we must assume the latter to be one-tenth of the former. This looks like too high a value, but it is at least interesting as a typical case for calculation. We find, instead of (124),

$$\log P = 0.48 + \tfrac{1}{2} \log \frac{g}{g_0} + \tfrac{3}{4} \log T - \tfrac{1}{2} \log (Gu_2 + Au_3) .$$

The calculation of P and N then proceeds as before, with very little more labor. The values of $\log P$ and $\log N$ for the main sequence are shown in Figure 6. N is now very nearly constant for the cooler stars, but diminishes considerably less for the hotter ones than before, while P increases rapidly during the ionization of the hydrogen. The increase in opacity, which on the former assumptions was due to both the electrons and the ions, is now thrown on the electrons alone, which therefore have to be more numerous.

The values of $\log L$ for $H\gamma$, Mg^+, and Fe^+ are shown in Figure 7, on both the new assumption and the old. The maximum for Fe^+ is hardly shifted at all, while those for $H\gamma$ and Mg^+ are shifted by

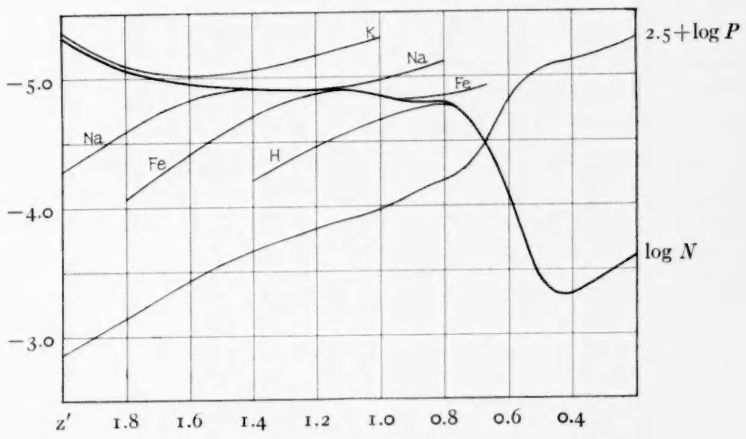
K
Na
Fe
Na
Fe
H
2.5+log P
log N
−5.0
−4.0
−3.0
z′ 1.8 1.6 1.4 1.2 1.0 0.8 0.6 0.4

FIG. 6

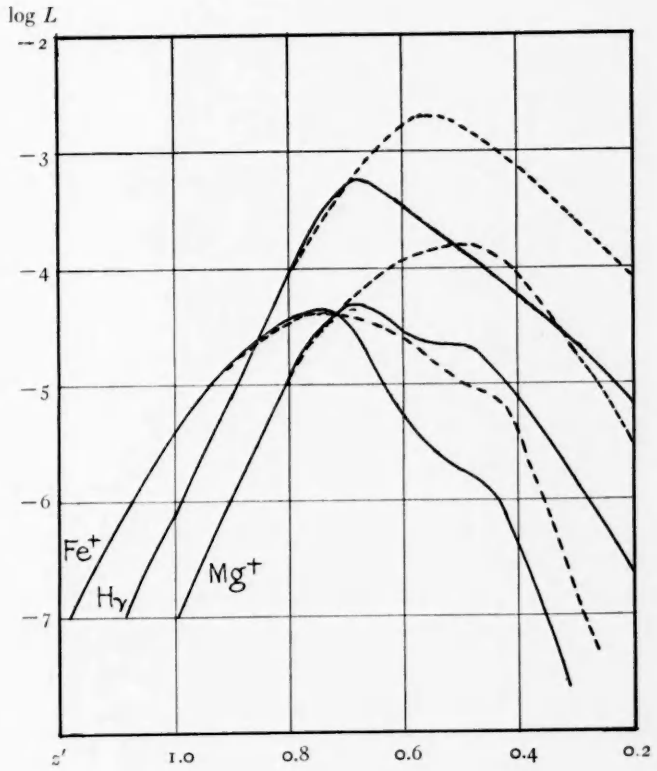
log L
−2
−3
−4
−5
−6
−7
Fe+
Hγ
Mg+
z′ 1.0 0.8 0.6 0.4 0.2

FIG. 7

-0.12 and -0.18, or by 0.22 and 0.33 spectral classes. This extreme assumption removes less than half the observed discordance. The rest may be attributed to distortion of the energy-curve. Curves for *Si*, *O*, and *He* in various stages of ionization for main-sequence stars are shown in Figure 8. The double maximum for Si^+ is produced by the ionization of hydrogen.

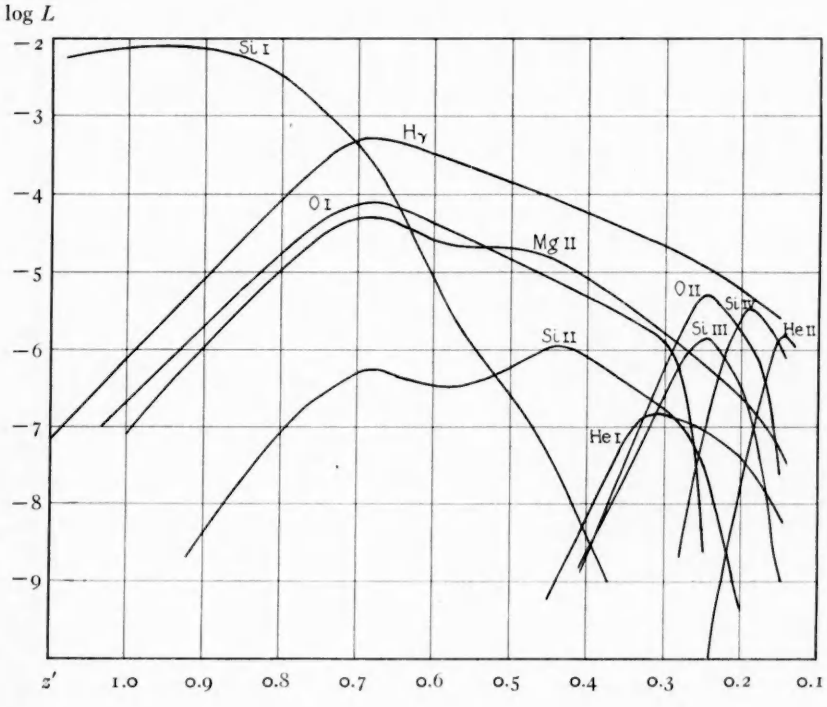

FIG. 8

In Table XVI the computed maxima are compared with Miss Payne's observations. The positions of the maxima agree closely with observation, which is not new. The predicted maximum intensities for Si^+, Si^{++}, Si^{+++}, and O^+ are nearly equal, as are the observed values. The *Si* line at λ 3905 is not so strong in proportion as might be expected; but the value of *f* for this line is uncertain. The results for *O* (λ 7772) are rough, since the values of *G* and *H* are different at this wave-length. The solar intensity of this line is 2; of

λ 4810 *Zn*, 3; and of λ 4481 Mg^+, 0. Since the values of log *L* for the three lines are −5.5, −5.2, and −5.8, the agreement is tolerable.

The abnormal strength of the lines of *H* and *He* appears to be confined to these elements. Measures of line contours and effective widths in spectra of classes A and B would be of great value in testing the theory.

27. The foregoing discussion leaves the maxima of stellar spectral lines in a satisfactory state. The assumption that the distortion of the energy-curve, which is known to exist, makes the color-temperatures of A stars too high removes the last serious discrepancy.

TABLE XVI

El.	*z'*	*T*	Pred. Sp.	Obs. Sp.	Obs. Int.	log *L*+10
Si	0.95	5300°	G2	Go	11.7	7.9
Hγ	.67	7500	A7	Ao	16.0	6.7
O	.68	7400	A8	.....	.....	5.9
Si$^+$	.68	7400	A8	Ao	9.3	3.7
	.43	11700	B8			4.0
He	.31	16200	B4	B3	11.8	3.2
Si^{++}	.25	20000	B3	B1	10.0	4.2
O$^+$	.24	21000	B2	B1	12.0	4.7
Si^{+++}	.19	26000	Bo	Bo	9.2	4.5
He$^+$	0.14	36000	O7:	O	>6	4.2

Two grave discordances between theory and observation remain. One, the abnormal strength of the lines of *H* and *He*, may be attributed with some confidence to the Stark effect, which is probably capable of accounting for it quantitatively. The other, and really the greater discordance, occurs in the spectra of red giants,[42] where the Balmer series, and the enhanced metallic lines as well, are enormously stronger than the present theory predicts. The Boltzmann factor has little influence in the latter case, and it seems to be impossible to account for the facts on the assumption of an isothermal atmosphere.

If, however, the opacity in the wave-length considered were much smaller than the Rosseland mean, we would receive light from deeper and hotter layers. An exact study of ionization and excitation in

[42] Adams and Russell, *Mt. Wilson Contr.*, No. 359; *Astrophysical Journal*, **68**, 27–35, 1928.

such an atmosphere, in which the radiant flux is not of black-body type, would be difficult. It is easy to see, however, that the number of highly excited atoms would be relatively increased. This effect may account for the curvature of the plots of log M against E found by Adams and the writer,[43] thus making the suspected departure from thermodynamic equilibrium a simple difference in temperature. Part of this effect, however, may be due to inaccuracies in the calibration of the very strong lines of low E in the red stars; but this circumstance cannot account for the behavior of the enhanced lines. Whether such a difference between the monochromatic and Rosseland opacities exists demands theoretical investigation. One would not expect to find it in stars where the maximum of the energy-curve occurs in the observed region, and there is little or no evidence of it there—the diagram given by Adams and the writer[44] being represented as well by straight lines as by curves. It is strongest in Betelgeuse and Antares, where the energy maximum is well in the infrared.

The increased pressure in the deeper layers inhibits ionization, while the high temperature favors it. A given percentage change in pressure produces the same effect at all temperatures, but a change in temperature is far more effective at low temperatures. The appearance of this effect in class M is therefore intelligible.

Present theory, however, indicates a high, rather than a low, opacity in the violet for such stars, and some other explanation may be preferable; for example, Unsöld's[45] suggestion that radiation in the far ultra-violet beyond the series limits of the metals may come through strongly enough to produce the excitation and ionization. A complete theory of stellar absorption lines will clearly be a complicated affair.

One more point may be mentioned in closing. Suppose that some stars contain a considerably less overwhelming excess of hydrogen than the average. If the difference extends to the interior, as well as the surface, these stars will be brighter than the mass-luminosity relation indicates; they will be, or tend to be, super-giants. The maximum of the Balmer lines will come at a higher temperature than

[43] *Ibid.*, p. 20.

[44] *Ibid.*, p. 23, Fig. 2.

[45] *Op. cit.*, 1, 1, 1930.

usual, but, owing to the low density, the lines will be sharper and may appear fainter than in normal stars. The lines of other elements will be stronger than usual, compared with hydrogen, especially the enhanced lines of the metals. The arc lines may be fainter, owing to the high ionization. Lines of high excitation, ordinarily absent, may appear.

These predictions of theory amount almost to a description of the spectrum of Alpha Cygni, and the suggestion that this and perhaps other c-stars are deficient in hydrogen appears plausible. To attribute the deficiency to partial exhaustion of hydrogen by processes of atomic synthesis in these very luminous stars is tempting.

PRINCETON, N.J.
May 2, 1933

NOTES

REMARKS ON DARK NEBULAE

ABSTRACT

An attempt is made to explain the shapes of dark nebulae. It is suggested that the larger particles of a nebula will concentrate near the stars, while the smaller particles will be repelled into more distant regions.

We have no definite indications of the size of particles forming the dark nebulae of Barnard and of Wolf. The criterion of mass required to produce the extinction of light observed in dark nebulae rules out gaseous matter as their chief constituent.[1] Debye's theory of the scattering of light by small particles (especially for non-transparent dielectrics) allows us to find this mass, given the extinction, the dimensions of the nebula, and the assumed size of the particles. I have performed such calculations, using Pannekoek's data on the obscuration in Taurus and allowing 10^5 as a rather conservative upper limit for the total mass of this cloud. The results are not definite. The cloud in question may be composed of particles of all sizes, from 7.10^{-7} to 10^{-1} cm in diameter, capable of producing all kinds of scattering (λ^{-4}, λ^{-1}, λ^0-laws); a minimum mass of approximately 15 corresponds to diameters of about 10^{-5} cm. The existing data on the color effects in dark nebulae are meager and somewhat ambiguous; further observations are greatly needed in order to decide the question. Nevertheless, even at present we can derive some conclusions as to the nature of dark nebulae by considering the forces acting on them.

In a recent paper[2] the writer has shown that on account of radiation pressure small ideally reflecting spheres producing scattering according to λ^{-4} or λ^{-1} cannot be retained as permanent satellites in

[1] Pannekoek, *Proceedings of the Amsterdam Academy*, **23**, 720, 1920; Eddington *Internal Constitution of the Stars*, p. 388.

[2] *Zeitschrift für Astrophysik*, **4**, 265, 1932.

the vicinity of supergiant or giant stars. For stars of classes Bo–B5, the lower limit of the radii of such particles turns out to be of the order of 0.2–0.04 cm, while late-type giants can retain particles of 10^{-3}–10^{-4} cm in diameter. On the other hand, dwarf stars exerting much smaller repulsive forces can possess, in the form of satellites, very small particles capable of producing selective scattering.[3] As to the order of magnitude, these results are also valid for non-transparent dielectric particles as well as for absorbing spheres of appreciable refractivity.

The different behavior of small particles in the vicinity of giant and dwarf stars seriously affects their distribution in galactic space. General as well as local fields of radiation and gravitation cause a special distribution of dark matter which could be calculated if these fields were known.

The properties of the general galactic field gradient represent the most difficult problem. At the present time we can only make some rough and indefinite guesses as to its direction and size. The accumulation of stars of high luminosity in and near the galactic plane gives some reason to think that there exists an appreciable general field gradient perpendicular to the galactic plane, tending, in higher latitudes, to drive away small particles. We have no knowledge concerning the projection of this gradient on the galactic plane, except assumptions by analogy based on the occurrence of dark belts in the equatorial planes of spiral nebulae. In contrast to the general galactic field, the local fields are more definite and can sometimes be inferred from the observational data.

The most outstanding examples of the repulsive action of stars of high luminosity on cosmic dust are the "cometary nebulae." These are feebly luminous, nebulous objects of comet-like shape with a more or less conspicuous star at the apex. At the present time the following nebulae of this kind are known:

N.G.C. 2261 (Hubble's variable nebula): Nucleus—R Monocerotis (spectrum nova-like). Nebular spectrum continuous.

[3] These calculations have been repeated recently by Schönberg and Jung (*Astronomische Nachrichten*, **247**, 413, 1933), who reached essentially the same conclusions.

N.G.C. 6729 (variable): Nucleus—R Coronae Australis (Gpe). Nebular spectrum continuous.

N.G.C. 2245: Nuclear spectrum B1. Nebular spectrum continuous.

N.G.C. 1555 (Hind's variable nebula): Nucleus—T Tauri (spectrum peculiar).

Anonyma ($\alpha = 6^h3^m$, $\delta = +18°42'$; 1920): Discovered by Hubble. Nuclear spectrum B1. Nebular spectrum continuous.

Anonyma: Nucleus CPD $-25°2286$. Spectrum Ma. Discovered by Van den Bos.

Anonyma: Nucleus B.D. $+28°645$ (RY Tauri, irregular variable, F8).

From a purely mechanical point of view, these cometary formations may be explained in much the same way as ordinary comets. A dust cloud containing particles of various sizes, small as well as large, if approached by a very luminous star, necessarily takes the shape of a comet, small particles being repelled, while the envelope of their orbits assumes the shape of a paraboloid of revolution. On the other hand, large particles (if present in sufficient quantity) are attracted by the star from the head, both head and tail shining with reflected starlight. Such a nebula is in general doomed to dissipation. In some cases, when the head exists and is sufficiently opaque, the large particles which compose it may produce a screening effect, partially protecting the smaller particles.

In four cases out of seven the nuclear stars are variables (two of them are nova-like). For purely physical reasons it is difficult to regard the variability of the nuclear stars as being caused by the clouds. On the other hand, nova-like outbursts would produce more or less symmetrical gaseous nebulosities. The phenomenon under consideration can be better explained in the following way: Stars showing nova-like spectra, and some irregular variables, are probably the most luminous objects in the sky. Brighter supergiants produce sharper cometary forms which are better illuminated and therefore easier to discover. The nuclear star, if seen through the axis of the nebula, would have a nebulous appearance. If the star is behind the nebula and the nebular tail, a color excess is to be expected and, as we know, is actually observed in many nebulous stars. If the nebula is optically thick and is seen edgewise, the nuclear star would be invisible and the nebula itself would be projected on the background stars as a dark, comet-like marking. Such probably, are Barnard's "dark comets," B64, etc.

We may present the problem in a somewhat different manner. Consider a group of early (or late supergiant) stars initially imbedded in a dense dust cloud composed of large as well as small particles. What will be the ultimate fate of such a cloud? We can expect that the larger particles will concentrate around individual stars as their Keplerian satellites, while the small particles will be dissipated, forming some kind of outer structureless obscuration. Such a steady state is probably attained in the Pleiades. On the other hand, under the protection of the large particles, if they form optically thick layers around the stars, small particles can be retained in a quasi-static state in the external parts of the nebula. If the steady state is not reached, these small particles can produce dark lanes and patches "radiating" from the central bright nebula which envelops the stars. As an example we may mention the dark and bright nebulae involving a group of early type stars near ρ Ophiuchi. These stars (ρ Ophiuchi, 22 Scorpii, B.D. $-24^\circ 12684$) are involved in bright nebulosities, showing continuous spectra with long dark lanes radiating from them in various selected directions.

If concentrated bright nebulosities enveloping the stars consist of large particles, as is here suggested, they cannot give anomalous color effects in their reflected light. Recent observations of relative color indices, star–nebula, did not reveal any differential color effect,[4] and thus corroborate these conclusions.

Another example of the repulsive effects may be suspected in B144. This splendid dark nebula has a parabolic shape; its axis, if prolonged on the celestial sphere, passes directly through γ Cygni (and through α Cygni farther on). The truncated summit of the nebula is 4° off γ Cygni; the nebula itself extends up to 4° in the opposite direction, fading out and diffusing into the regular stellar background. Assuming that for γ Cygni (F8p), according to a determination made at Victoria, $M = -3^m$, we find the distance of the nebular summit from the star to be about 7 parsecs (if the axis of the nebula is tangential to the celestial sphere).

If the local field is not central but results from several dispersed stars, its effect on a dark nebula should be very complicated; in this

[4] O. Struve, C. T. Elvey, and P. C. Keenan, *Astrophysical Journal*, **77**, 274, 1933.

case the equipotential surfaces can take any imaginable shape. If such a field is repulsive, its strength would be proportional to $e^{-\tau}r^{-2}$, τ being the optical depth reckoned from the outer boundary. Such a field would result in a one-sided compression of the cloud, producing one sharp boundary, the opposite boundary being diffuse. Every student of dark nebulae notices a remarkable property of these objects which has been described by Barnard as follows: "In many cases one side of a dark marking is very definite, while the other side is diffused. This occurs so often that there must be some reason for it."[5] Striking examples are B50, 143, 160, etc. It may be concluded, therefore, that in some cases the outer boundaries of dark nebulae represent equipotential surfaces, while in other cases they can be interpreted as the envelopes of trajectories or lines of flow.

It goes without saying that the action of repulsive forces can explain only part of the observed contours of dark nebulae. It is probable, however, that dark nebulae represent rather ephemeral formations whose existence and evolution are determined chiefly by the local field of radiation and gravitation produced by stars. Internal forces are of minor importance.

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August 1933

VARIABILITY OF BRIGHT *Hα* IN α CYGNI

ABSTRACT

The emission line of *Hα* in Deneb is found to vary in intensity.

Some time ago we found that bright *Hα* in the spectrum of Rigel (β Orionis) varies in intensity. Since this line resembles in structure the same line in α Cygni, we decided to observe the latter with our grating spectrograph, which is used in connection with the 69-inch reflector of the Perkins Observatory. The plates indicate a large variation in the intensity of the bright line, which may possibly be

[5] *Ibid.*, **49**, 1, 1919.

associated with a variation of the absorption lines. The period may be in the neighborhood of several months, although a short period is not impossible.

Date	Descrip. of *Hα* Emiss.	Qual.	Date	Descrip. of *Hα* Emiss.	Qual.
1933 Mar. 9.46...	Weak	p	1933 Aug. 22.15...	Weak	f
Apr. 29.38...	Fair	p	Aug. 24.06...	Weak	g
May 18.38...	Strong	g	Aug. 26.08...	Fair	g
May 30.38...	Fair	f	Aug. 26.19...	Fair	g
June 15.37...	Fair	g	Aug. 28.10...	Fair	g
July 31.14...	Absent	g	Aug. 29.17...	Fair	g
Aug. 19.18...	Weak	f	Sep. 2.06...	Strong	g
Aug. 20.06...	Weak	f	Sep. 7.05...	Fair	f
Aug. 21.06...	Fair	g	Sep. 7.10...	Weak	p
Aug. 21.12...	Fair	g	Sep. 18.08...	Very strong	g
Aug. 21.18...	Fair	g	Sep. 28.06...	Strong	p
Aug. 22.06...	Fair	f	Oct. 5.05...	Strong	f
			Oct. 12.01...	Fair	g

The emission line was practically absent on an excellent plate taken on July 31 and was very strong on an equally satisfactory plate taken on September 18. On the last plate the absorption line of *Hα* is split in two by a narrow emission line which is much weaker than the emission line on the red side of the absorption line.

O. STRUVE
F. E. ROACH

October 13, 1933

REVIEWS

Atom and Cosmos. By HANS REICHENBACH. Translated by EDWARD S. ALLEN. New York: Macmillan Co., 1933. Pp. 300. Bound, $2.00.

In this book, which is an enlargement of a series of radio lectures delivered from Berlin some three years ago, the author sets himself the task of presenting to the layman an account of the important advances which physics has made in the twentieth century, and of the changes resulting therefrom in the basic concepts of scientific thought.

In general, the material is presented as simply and as precisely as could be expected in a strictly non-mathematical work. The author treats in his first section the concepts of non-Euclidian space and relativity of motion, together with their astronomical significance. Following this he discusses radiation, its apparent dual nature as wave-train and stream of corpuscles, and the reconciliation of these two theories in the developments of the last ten years. In the next section, various pictures of matter as made up of molecules, atoms, or electrons and protons are given. The author speaks, too, of the early pictures by Rutherford and Bohr of the interior of the atom. The last chapter of this section, on "The Wave Character of Matter," is a fitting introduction to that which follows, in which the author speaks in everyday terms of the concept of probability, of Heisenberg's uncertainty principle, and of the difficulties concerning fundamental concepts and world-pictures which scientific thought of the present century has had to face.

Whether the author's prefatory remark that his presentation "does not presuppose any knowledge of the kind taught in schools" may be taken literally is doubtful; however, the book does accomplish its aim, to give "the physicist's way of thinking, and a general view of the results of his research."

PAUL RUDNICK